Numerical solution of continuous-time DSGE models under Poisson uncertainty*

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Abstract

We propose a simple and powerful method for determining the transition process in continuous-time DSGE models under Poisson uncertainty numerically. The idea is to transform the system of stochastic differential equations into a system of functional differential equations of the retarded type. We use the Waveform Relaxation algorithm to provide a guess of the policy function and solve the resulting system of ordinary differential equations by standard methods and fix-point iteration. Analytical solutions are provided as a benchmark from which our numerical method can be used to explore broader classes of models. We illustrate the algorithm simulating both the stochastic neoclassical growth model and the Lucas model under Poisson uncertainty which is motivated by the Barro-Rietz rare disaster hypothesis. We find that, even for nonlinear policy functions, the maximum (absolute) error is very small.

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1 Introduction

Background. Since the seminal paper of Brock and Mirman (1972), presenting a stochastic neoclassical one-sector growth model, the dynamic stochastic general equilibrium (DSGE) models have become the workhorse in dynamic macroeconomic theory. This benchmark economy gave rise to the development of advanced tools for capturing the main features of aggregate fluctuations and for policy recommendations.

The literature on DSGE models, however, has been surprisingly quiet on the effects of large economic shocks such as natural disasters and financial and/or economic crises. Rare events in the form of natural disasters or technological improvements play an important role in the consumption-saving decision of economic agents: Barro (2006, 2009) finds that rare disasters are sufficiently frequent to have substantial asset pricing and welfare implications; rare events in the form of technological changes are found in quality ladder and matching models (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Lentz and Mortensen, 2008), models explaining both growth and economic fluctuations (Wälde, 1999, 2005), and models of investment under uncertainty with regime shifts (Guo, Miao and Morellec, 2005).

In the field of macroeconometrics, Posch (2009b) finds empirical evidence for Poisson jumps in US macro data estimating continuous-time DSGE models.[1] One caveat of non-linear and/or non-normal models usually is the lack of analytical solutions. This is unfortunate, so the literature is making a huge effort in developing powerful computational methods, extending the perturbation and projection procedures (Judd, 1992; Judd and Guu, 1997).[2] Although most numerical methods generally are locally highly accurate, the effects of large economic shocks, such as rare disasters on approximation errors, are largely unexplored.

The open question. How does the presence of rare events affect the optimal decisions in dynamic general equilibrium economies? It seems important to understand the implications of the simple awareness of the possibility for large economic shocks on optimal decisions, such as consumption or leisure, for the aggregate dynamics in DSGE models. In particular, we need to relax parametric restrictions which give rise to closed-form solutions. Since current computational methods largely consider uncertainty as emerging from small shocks, it does not come as a surprise that no in-depth analysis has been provided to date.

Our message. This paper proposes a simple and powerful method for determining the transition process in continuous-time DSGE models under Poisson uncertainty numerically.

[1] Related studies find that non-linearities and/or non-normalities are important features of US aggregate data (cf. Fernández-Villaverde and Rubio-Ramírez, 2007; Justiniano and Primiceri, 2008).

[2] Many studies compare computational methods for DSGE models (cf. Taylor and Uhlig, 1990; Christiano and Fisher, 2000; Schmitt-Grohé and Uribe, 2004; Aruoba, Fernández-Villaverde and Rubio-Ramírez, 2006; Caldara, Fernández-Villaverde, Rubio-Ramírez, and Yao, 2009).

We show how to extend existing standard algorithms when we allow for the possibility of rare events. We propose the continuous-time formulation of DSGE models to circumvent the difficulties that arise when we allow for departures from normalities in the form of Poisson shocks in non-linear models, which is largely unexplored land.

Our framework. Our analysis builds on the continuous-time formulation of a stochastic neoclassical growth model based on Merton (1975). This procedure is applicable to models which imply an equilibrium system of controlled stochastic differential equations (SDEs) under Poisson uncertainty where the controls are Markov controls in the form of policy functions (cf. Sennewald, 2007). There is no conceptional difficulty to allow for stochastic processes with controlled variance-covariance and/or controlled jump sizes.

We use the continuous-time formulation of DSGE models for two reasons. Firstly, we can easily compute stochastic differentials for transformations based on random variables under Poisson uncertainty. Secondly, for parametric restrictions we can solve the models by hand and obtain closed-form policy functions which can be used as a point of reference. From this benchmark numerical methods can explore broader classes of models (cf. Judd, 1997).

Our idea is to transform the system of SDEs into a system of functional differential equations of the retarded type (RDEs). We then use the Waveform Relaxation algorithm to provide a guess of the policy function and solve the resulting system of ordinary differential equations (ODEs) by standard methods and fix-point iteration.

Results. Our solution method works. Although the suggested procedure computes the policy functions for the complete state space - even for non-linear solutions - the maximum (absolute) error compared to the exact solutions is very small. A strength of our approach is that existing algorithms can be extended to allow for Poisson uncertainty. We illustrate our approach for two popular methods computing numerical solutions to dynamic general equilibrium models, i.e., the backward integration (Brunner and Strulik, 2002) and the Relaxation algorithm (Trimborn, Koch and Steger, 2008). From an economic point of view, we find that (potential) large shocks affect optimal consumption and hours strategies.

Table of contents. The paper proceeds as follows. Section 2 holds the class of macro models to which the algorithm is applicable. Section 3 relates the models to our numerical solution technique based on Waveform Relaxation. Section 4 gives examples and computes the numerical errors for the benchmark solutions. Section 5 concludes.

2 The macroeconomic theory

This section introduces a broad class of macro models which can be solved by means of Waveform Relaxation. For illustration, we formulate the models for scalar processes, i.e., assuming only one control and one state variable. As shown below, the class of models can be extended to the case of multiple controls and states without any conceptional difficulties.

Consider the following typical autonomous infinite horizon stochastic control problem,

$$\max E \int_0^\infty e^{-\rho t} u(x_t, c_t) dt \quad s.t. \quad dx_t = f(x_t, c_t)dt + g(x_{t-}, c_{t-})dN_t, \tag{1}$$
$$x_0 = x, \; N_0 = z, \; (x, z) \in U_x \times \mathbb{R}_+, \; U_x \subseteq \mathbb{R},$$

where $\{N_t\}_{t=0}^\infty$ is a Poisson process with arrival rate $\lambda = \lambda(x_t, c_t)$, and a value $x_{t-} \equiv \lim_{s \to t} x_s$ $s < t$ denotes the left-limit of this variable at time t. Intuitively, x_{t-} denotes the value of the variable an instant before a possible jump, such that for continuous paths $x_t = x_{t-}$.

2.1 Bellman's principle and reduced form descriptions

Closely following Sennewald (2007), choosing an admissible control, $c \in U_c \subseteq \mathbb{R}$, from the control region U_c and using $V(x)$ as the value function, we obtain the *Bellman equation*

$$\rho V(x) = \max_{c \in U_c} \left\{ u(x, c) + \frac{1}{dt} E_0 dV(x) \right\},$$

which is a necessary condition for optimality. Using Itô's formula (change of variables),

$$dV(x) \;\; = \;\; f(x, c)V_x(x)dt + (V(x + g(x, c)) - V(x))dN_t.$$

If we take the expectation of the integral form and use the martingale property, assuming that the above integrals exist, in particular that $u(x_t, c_t)$ satisfies some boundedness condition (Sennewald, 2007, Theorem 2), we arrive at

$$E_0 dV(x) \;\; = \;\; f(x, c)V_x(x)dt + (V(x + g(x, c)) - V(x))\lambda(x, c)dt, \tag{2}$$

and the Bellman equation becomes

$$\rho V(x) = \max_{c \in U_c} \{ u(x, c) + f(x, c)V_x + (V(x + g(x, c)) - V(x))\lambda(x, c) \}.$$

A neat result about the continuous-time formulation (compared to discrete-time models) is that the Bellman equation (2) is, in effect, a deterministic differential equation because the expectation operator disappears (Chang, 2004, p.118). The *first-order condition* reads

$$\begin{aligned} & u_c(x, c) + f_c(x, c)V_x(x) + V_x(x + g(x, c))g_c(x, c)\lambda(x, c) \\ & \qquad + (V(x + g(x, c)) - V(x))\lambda_c(x, c) = 0, \end{aligned} \tag{3}$$

for any $s = t \in [0, \infty)$ making the optimal control a function of the state variable, $c_t = c(x)$. In contrast to deterministic control problems we obtain a first-order term linking both the utility before and after a jump resulting from the Poisson process (controlled jump size), and the value of the optimal program before and after the jump (controlled jumps). Note that the costate variable and the value function is evaluated at different values of x, respectively.

For the *evolution of the costate* we use the maximized Bellman equation,

$$\rho V(x) = u(x, c(x)) + f(x, c(x))V_x + (V(x + g(x, c(x))) - V(x))\lambda(x, c(x)),$$

where the optimal control is a function of the state variables. We make use of the envelope theorem to compute the costate,

$$\begin{aligned}
\rho V_x(x) \;&=\; u_x(x, c(x)) + f_x(x, c(x))V_x + f(x, c(x))V_{xx} \\
&\quad +(V_x(x + g(x, c(x)))(1 + g_x(x, c)) - V_x(x))\lambda(x, c(x)) \\
&\quad +(V(x + g(x, c(x))) - V(x))\lambda_x(x, c(x)).
\end{aligned}$$

Collecting terms we obtain

$$\begin{aligned}
(\rho - f_x(x, c) + \lambda(x, c))\, V_x \;&=\; u_x(x, c) + f(x, c)V_{xx} + V_x(x + g(x, c))(1 + g_x(x, c))\lambda(x, c) \\
&\quad +(V(x + g(x, c)) - V(x))\lambda_x(x, c). \qquad (4)
\end{aligned}$$

Using Itô's formula, the costate obeys

$$dV_x(x) \;=\; (f(x, c)V_{xx}(x))\, dt + (V_x(x + g(x, c)) - V_x(x))\, dN_t,$$

where inserting (4) yields the evolution of the costate variable

$$\begin{aligned}
dV_x(x) \;&=\; ((\rho - f_x(x, c) + \lambda(x, c))\, V_x(x) - u_x(x, c) - V_x(x + g(x, c))(1 + g_x(x, c))\lambda(x, c))\, dt \\
&\quad -(V(x + g(x, c)) - V(x))\lambda_x(x, c) + (V_x(x + g(x, c)) - V_x(x))\, dN_t.
\end{aligned}$$

As the final step, we implicitly obtain the *Euler equation* from the first-order condition.[3] Further, the transversality condition may be formulated as $\lim_{t\to\infty} e^{-\rho t}V(x) \geq 0$ for all admissible paths, where the equality holds for the optimal solution.

Case 2.1 (Controlled SDE) *Consider the stochastic control problem where* $g(x, c) = g(x)$, $\lambda(x, c) = \lambda$, $u(x, c) = u(c)$ *and* $f_c(x, c) = -1$.

Observe that the first-order condition (3) reduces to $u'(c_t) = V_x(x_t)$. Hence, a reduced form is given by the following system of SDEs for the state (1) and the costate,

$$\begin{aligned}
du'(c(x_t)) \;&=\; ((\rho - f_x(x_t, c_t) + \lambda)\, u'(c_t) - u'(c(x_t + g(x_t)))(1 + g_x(x_t))\lambda)\, dt \\
&\quad + (u'(c(x_{t-} + g(x_{t-}))) - u'(c(x_{t-})))\, dN_t. \qquad (5)
\end{aligned}$$

[3]This 'three-step' procedure was suggested in Sennewald and Wälde (2006).

Suppose $u''(c) \neq 0$, then for $c = h(u'(c))$ where $h(\cdot)$ is the inverse function, it reads

$$\begin{aligned} dc_t &= \left((\rho - f_x(x_t, c_t) + \lambda)\, u'(c_t) - u'(c(x_t + g(x_t)))(1 + g_x(x_t))\lambda\right) / u''(c_t) dt \\ &\quad + \left(c(x_{t-} + g(x_{t-})) - c(x_{t-})\right) dN_t, \qquad (6a) \\ dx_t &= f(x_t, c_t)dt + g(x_{t-})dN_t, \qquad (6b) \end{aligned}$$

where we used the property of the inverse function, $dh(u'(c))/du'(c) = dc/du'(c) = 1/u''(c)$.

Case 2.2 (Controlled SDE with controlled momentum) *Consider the stochastic control problem where* $\lambda(x, c) = \lambda$, $u(x, c) = u(c)$, $g_c(x, c) \neq 0$, *and* $f_c(x, c) = -1$.

Observe that the first-order condition (3) reduces to $u'(c) - V_x(x) + V_x(x + g(x, c))g_c(x, c)\lambda$. Hence, a reduced form is given by the following system of SDEs,

$$\begin{aligned} dV_x &= \left((\rho - f_x(x_t, c_t) + \lambda)\, V_x(x_t) - V_x(x_t + g(x_t, c_t))(1 + g_x(x_t, c_t))\lambda\right) dt, \\ &\quad + \left(V_x(x_{t-} + g(x_{t-}, c_{t-})) - V_x(x_{t-})\right) dN_t \qquad (7a) \\ dx_t &= f(x_t, c_t)dt + g(x_{t-}, c_{t-})dN_t, \qquad (7b) \end{aligned}$$

using the first-order condition as an algebraic equation.

Case 2.3 (Controlled SDE with controlled arrivals) *Consider the stochastic control problem where* $g(x, c) = g(x)$, $\lambda(x, c) = \lambda(c)$, $u(x, c) = u(c)$ *and* $f_c(x, c) = -1$.

Observe that the first-order condition (3) reduces to $u'(c) - V_x(x) + (V(x + g(x, c)) - V(x))\lambda_c(c) = 0$. Hence, a reduced form is given by the following system of SDEs,

$$\begin{aligned} dV &= f(x_t, c_t)V_x(x_t)dt + (V(x_t + g(x_{t-})) - V(x_{t-}))dN_t, \qquad (8a) \\ dV_x &= \left((\rho - f_x(x_t, c_t) + \lambda(c_t))\, V_x(x_t) - V_x(x_t + g(x_t, c_t))(1 + g_x(x_t))\lambda(c_t)\right) dt \\ &\quad + \left(V_x(x_{t-} + g(x_{t-})) - V_x(x_{t-})\right) dN_t, \qquad (8b) \\ dx_t &= f(x_t, c_t)dt + g(x_{t-})dN_t, \qquad (8c) \end{aligned}$$

using the first-order condition as an algebraic equation.

3 The numerical solution

This section transforms the reduced forms into a system of functional differential equations of the retarded type (RDEs). We relate the problem to Waveform Relaxation, providing a guess of the optimal policy function and solving the resulting systems of ODEs.

3.1 Description of the problem

In order to illustrate our solution method, we focus on a system of controlled stochastic differential equations (6) of Case 2.1, which can be generalized to

$$dc_t = f_1(c_t, x_t, \vec{c}, \vec{x})\,dt + g_1(c_{t-}, x_{t-})dN_t, \tag{9a}$$

$$dx_t = f_2(c_t, x_t, \vec{c}, \vec{x})\,dt + g_2(c_{t-}, x_{t-})dN_t, \tag{9b}$$

given initial states x_0. The generalization stems from the fact that the system of SDEs in (9) at date t implicitly depends on its complete solution path $\{\vec{c}, \vec{x}\} \equiv \{(c_\tau, x_\tau), \tau \in \mathbb{R}\}$, i.e., on the optimal pair at any admissible date τ. Technically, we define the functions $\vec{c} : \mathbb{R} \to U_c \subseteq \mathbb{R}^{n_c}$ and $\vec{x} : \mathbb{R} \to U_x \subseteq \mathbb{R}^{n_x}$, where U_c denotes the control region and U_x the state space with n_c and n_x denoting the number of controls and states, respectively. Hence, we define the functions f_1, f_2, g_1, g_2 as

$$f_1 : \quad U_c \times U_x \times C^k(\mathbb{R}, U_c) \times C^k(\mathbb{R}, U_x) \to \mathbb{R}^{n_c}, \tag{10a}$$

$$f_2 : \quad U_c \times U_x \times C^k(\mathbb{R}, U_c) \times C^k(\mathbb{R}, U_x) \to \mathbb{R}^{n_x}, \tag{10b}$$

$$g_1 : \quad U_c \times U_x \to \mathbb{R}^{n_c}, \tag{10c}$$

$$g_2 : \quad U_c \times U_x \to \mathbb{R}^{n_x}, \tag{10d}$$

which are assumed to be sufficiently smooth. More precisely, all functions are of class C^k, i.e., the partial derivatives of its component functions of up to (and including) order k exist and are continuous, where k is sufficiently large. In Case 2.1 (controlled SDE), the dependency on the complete solution stems from the observation that a Poisson jump implies that the state variable x_t jumps by $g_2(c_{t-}, x_{t-})$. Consumers consider the possibility of jumps in their optimal control problem and optimally take into account the level of their control variables if such rare events occur. Therefore at one point in time, say t, the solution at another point in time $\tau \in \mathbb{R}$ (or another given state x_τ) influences the slopes dc_t and dx_t.

System (9) has to be augmented by boundary conditions for the beginning and the end of the time horizon. Transversality conditions usually require (scale-adjusted) variables to converge towards some interior steady states for $t \to \infty$, conditional on no jumps, $dN_t \equiv 0$.[4] We denote steady-state values by $\{c^*, x^*\} \subseteq \{\vec{c}, \vec{x}\}$. However, it is not sufficient to compute the solution on the domain $[x_0, x^*]$, because a state could be thrown back to an even smaller value than x_0 or jump to a value above x^*. In that perspective, the optimal control on $[x_0, x^*]$ depends on the optimal control for some $x_t < x_0$ and $x_t > x^*$. Since this argument holds for any component of the state vector in the state space U_x, the solution has to be computed on the entire domain U_x, which for macroeconomic problems usually is $U_x = \mathbb{R}_+^{n_x}$.

[4] If no ambiguity arises, we use 'steady state' and 'conditional steady state' interchangeably.

We assume that system (9) has a unique solution, $\{\vec{c}, \vec{x}\}$, which *only* depends on state variables, but not on the random variable N_t. This means that given the actual state x_t at date t, the optimal control variables are uniquely determined. For cases where one function, say $f_1 = f_1(c_t, x_t, Z_t, \vec{c}, \vec{x})$ is a function of a random variable, Z_t, in general our procedure requires conditioning, $Z_t = z$, such that $f_1 = f_1(c_t, x_t, z, \vec{c}, \vec{x}) \equiv \bar{f}_1(c_t, x_t, \vec{c}, \vec{x})$.

3.2 The Waveform Relaxation algorithm

The crucial task for the numerical solution is to compute the policy function implied by the (conditional) deterministic system, i.e., for $dN_t \equiv 0$,

$$dc_t = f_1(c_t, x_t, \vec{c}, \vec{x})\, dt, \tag{11a}$$

$$dx_t = f_2(c_t, x_t, \vec{c}, \vec{x})\, dt. \tag{11b}$$

In a second step, the stochastic paths are obtained by adding the Poisson process N_t, making use of the entire solution path $\{\vec{c}, \vec{x}\}$ and thus $c_t = c(x_t)$. The controls and the states follow the paths implied by the system (11) as long as no jump occurs (pathwise continuous). If a jump occurs at date t_-, the systems adjusts according to g_1 and g_2 with $c_{t-} = c(x_{t-})$.

In the mathematical literature, the equations in system (11) are referred to as functional differential equations of the retarded type (cf. Hale, 1977; Kolmanovskii and Myshkis, 1999). In this special case the 'delay' depends on the unknown solution, because a jump changes the value of the state variable and throws the economy 'back in time' to a different state. Here, the discrete change is known in terms of the state, not in terms of time, which however does not preclude us to apply a similar solution method for our autonomous problem.

For calculating the policy function $c_t = c(x_t)$ we exploit the fact that numerous numerical methods are available to solve (11) *without* a dependency on the optimal solution,

$$dc_t = \tilde{f}_1(c_t, x_t)\, dt, \tag{12a}$$

$$dx_t = \tilde{f}_2(c_t, x_t)\, dt. \tag{12b}$$

The idea of Waveform Relaxation algorithms is as follows: by providing a guess of the optimal pair $\{\vec{c}_0, \vec{x}_0\}$, system (11) reduces to (12), because the feedback of the solution path on dc_t and dx_t is neglected.[5] Now, problem (12) is a standard system of ODEs and can thus be solved by standard algorithms.[6] In general, the obtained solution $\{\vec{c}_1, \vec{x}_1\}$ will be different

[5] Waveform Relaxation algorithms for initial value problems and appropriate error estimation are described in Feldstein, Iserles and Levin (1995), Bjørhus (1994) and Bartoszewski and Kwapisz (2001). Alternative procedures for solving system (11) are collocation methods as described in Bellen and Zennaro (2003).

[6] For problems with one state variable, among others, these are the backward integration procedure (Brunner and Strulik, 2002) and the procedure of time elimination (Mulligan and Sala-i-Martín, 1991). For problems with multiple state variables we can use projection methods (e.g., Judd, 1992), the method of Mercenier and Michel (1994), and the Relaxation method (Trimborn et al., 2008).

from the initial guess $\{\vec{c}_0, \vec{x}_0\}$. Hence, a solution of the original deterministic problem (11) is not found yet. In the next step the initial guess is updated to $\{\vec{c}_1, \vec{x}_1\}$ and the loop is repeated. If the updated solution $\{\vec{c}_i, \vec{x}_i\}$ is the same as the guess $\{\vec{c}_{i-1}, \vec{x}_{i-1}\}$, a solution of the deterministic problem (11) is found and thus $c_t = c(x_t)$.

More formally, we construct a fix-point iteration for the operator $\mathcal{N}$ such that a function z is a fix point of this operator: $\mathcal{N}(z) = z$. The function z represents the desired solution, $z : \mathbb{R} \to \mathbb{R}^{n_c+n_x}$. The operator $\mathcal{N}$ is defined by a modification of problem (11). We start with a trial solution z_0 and iterate by evaluating $\mathcal{N}$, until $\|z_i - z_{i-1}\|$ is sufficiently small.

For defining the operator $\mathcal{N}$, we take a trial solution $\{\vec{c}_0, \vec{x}_0\}$ as given. We define

$$dc_i = f_1\left(c_i(t), x_i(t), \vec{c}_{i-1}, \vec{x}_{i-1}\right) dt, \tag{13a}$$

$$dx_i = f_2\left(c_i(t), x_i(t), \vec{c}_{i-1}, \vec{x}_{i-1}\right) dt, \tag{13b}$$

for each iteration $i = 1, \ldots, n$. Hence, system (13) represent a system of ordinary differential equations which can be solved by the existing standard numerical methods.

For single-state problems ($n_x = 1$) we employ the backward integration method proposed by Brunner and Strulik (2002).[7] For the description of this method, recall that equations (13a) and (13b) represent a system of ODEs with an interior, computable stationary point. This point usually exhibits a saddle-point structure, i.e., a stable one-dimensional manifold (policy function) connecting the steady state to the origin, and an unstable one-dimensional manifold. Our task is to compute the stable manifold numerically, for which we exploit the saddle-point structure. By reversing time, the stable (unstable) manifold becomes an unstable (stable) manifold. Thus, by starting near the manifold, solution trajectories are attracted by the policy function.

An important difference to standard methods in each iteration step is the evaluation of $\{\vec{c}_i, \vec{x}_i\}$. The reason is that the solution of the previous iteration $\{\vec{c}_{i-1}, \vec{x}_{i-1}\}$ is only available on a mesh of points in time, or equivalently that the function $c_{i-1}(x_{i-1})$ in the phase space is only represented at certain points. However, functions f_1 and f_2 of system (13) also need an evaluation of $\{\vec{c}_{i-1}, \vec{x}_{i-1}\}$ at interior points. We employ a cubic spline interpolation of $c_{i-1}(x_{i-1})$ to evaluate f_1 and f_2. In order to evaluate the improvement in convergence, a suitable norm has to be chosen. We calculate the deviation of the policy function between two iterations on a mesh of points representing the whole state space $(0, \infty)$ and employ the Euclidian norm, i.e., $||c_i(x_i) - c_{i-1}(x_i)||$, where $0 < x_i(1) < ... < x_i(M) < \infty$ and M denotes the number of points on the mesh (M determines the accuracy of the solution).

For multiple-state problems ($n_x > 1$) we employ the Relaxation algorithm as described in Trimborn et al. (2008) to solve the deterministic system (13). This method can be applied

[7]Note that backward iteration can be applied to any number of control variables, i.e., $n_c \geq 1$.

to continuous-time deterministic problems with any number of state variables. The principle of relaxation is to construct a large set of non-linear equations, the solution of which represents the desired trajectory. This is achieved by a discretization of the involved differential equations on a mesh of points in time. The set of differential equations is augmented by algebraic equations representing equilibrium conditions or (static) no-arbitrage conditions at each mesh point. Finally, equations representing the initial and final boundary conditions are appended. The whole set of equations is solved simultaneously.

For multiple-state problems, the policy function is also multidimensional. Technically, $c_t = c(x_t) : U_x \subseteq \mathbb{R}^{n_x} \to U_c \subseteq \mathbb{R}^{n_c}$. We select starting values x_0 uniformly located in a rectangle in the state space U_x and calculate transitional dynamics starting from each of these initial values. The solutions give a good representation of the policy function. Again, the policy function is only available on a mesh in the state space. Similar to the simulations with a one-dimensional state space, we employ a cubic spline interpolation to obtain the policy function at arbitrary interior points. Different from the procedure above, we use only the initial value of each iteration for interpolation. This turns out to be a more robust approach, presumably due to the evenly spaced grid one obtains in this case.

4 Examples

The following examples are intended to illustrate potential economic applications in macro. To start with, we first consider the stochastic Ramsey problem with a single control and state variable, and then use a stochastic version of the Lucas model of endogenous growth mainly to illustrate the fact that multi-dimensional systems do not pose conceptional difficulties. In order to keep notation simple, we only consider problems faced by a benevolent planner, and use capital letters to denote variables in the planning problem which correspond to individual variables in the household's and firms' problems.

4.1 A neoclassical growth model with disasters

This section solves the stochastic neoclassical growth model under Poisson uncertainty which is motivated by the Barro-Rietz rare disaster hypothesis (Rietz, 1988; Barro, 2006).

Specification. Suppose that production takes the form of Cobb-Douglas, $Y_t = K_t^{\alpha} L^{1-\alpha}$, $0 < \alpha < 1$. Labor is supplied inelastically and capital can be accumulated according to

$$dK_t = (Y_t - C_t - \delta K_t)\, dt - \gamma K_{t-} dN_t, \quad K_0 = x,\ N_0 = z, \quad (x, z) \in \mathbb{R}_+^2, \quad 0 < \gamma < 1, \tag{14}$$

where N_t denotes the number of (natural) disasters up to time t, occasionally destroying γ

percent of the capital stock K_t at an arrival rate $\lambda \geq 0$.[8]

The benevolent planner maximizes welfare by choosing controls $c \in U_c \subseteq \mathbb{R}_+$,

$$\max E \int_0^\infty e^{-\rho t} u(C_t) dt \quad \text{where} \quad u' > 0, \; u'' < 0 \quad s.t. \quad (14). \tag{15}$$

Solution. From the Bellman principle, a necessary condition for optimality is

$$\rho V(x) = \max_{c \in U_c} \left\{ u(c) + (x^\alpha L^{1-\alpha} - c - \delta x) V_x + (V(x - \gamma x) - V(x))\lambda \right\}, \tag{16}$$

and the first-order condition corresponding to (3) reads

$$u'(c) - V_x(x) = 0 \tag{17}$$

for any $s = t \in [0, \infty)$, making the control variable a function of the state variable, $c = c(x)$. Hence, the problem (15) can be summarized as a system of controlled SDEs,

$$\begin{aligned} dC_t &= \big((\rho - \alpha K_t^\alpha L^{1-\alpha} - \delta + \lambda) u'(C_t)/u''(C_t) - u'(C((1-\gamma)K_t))/u''(C_t)(1-\gamma)\lambda\big)\, dt \\ &\quad + (C((1-\gamma)K_{t-}) - C(K_{t-}))\, dN_t, && \text{(18a)} \\ dK_t &= (K_t^\alpha L^{1-\alpha} - C_t - \delta K_t) dt - \gamma K_{t-} dN_t, && \text{(18b)} \end{aligned}$$

corresponding to system (6) in Case 2.1.

Case 4.1 (CRRA preferences) *Consider the case where $-u''(C_t)C_t/u'(C_t) = \theta$.*

For constant relative risk aversion (CRRA) where $-u''(C_t)C_t/u'(C_t) = \theta$, we obtain

$$\begin{aligned} dC_t &= \big(\alpha K_t^\alpha L^{1-\alpha} - \rho - \delta - \lambda + \lambda(1-\gamma)\tilde{C}(K_t)^{-\theta}\big) C_t/\theta dt - \big(1 - \tilde{C}(K_{t-})\big) C_{t-} dN_t, \\ dK_t &= (K_t^\alpha L^{1-\alpha} - C_t - \delta K_t) dt - \gamma K_{t-} dN_t, \end{aligned}$$

where $\tilde{C}(K_t) \equiv C((1-\gamma)K_t)/C(K_t)$, such that $1 - \tilde{C}(K_{t-})$ denotes the percentage drop of optimal consumption after a disaster. CRRA utility is frequently used in macroeconomics, thus our numerical results all employ this standard description of preferences.

4.1.1 Evaluation of the algorithm

We calculate numerical solutions for two benchmark calibrations. In both cases, an analytical representation of the policy function can be computed for plausible parameter restrictions.

[8] For a stochastic neoclassical growth model with elastic labor supply and the asset market implications of the Barro-Rietz rare disaster hypothesis, the interested reader is referred to Posch (2009a).

Therefore, we can compare numerical and analytical solutions and calculate computational errors to evaluate the performance of the Waveform Relaxation algorithm.[9]

Because the neoclassical growth model has one state variable, it is well suited for the backward integration procedure (cf. Brunner and Strulik, 2002). As explained above, by starting near the steady-state value K^* (the value towards which the economy tends if no disasters occur), the solution trajectories are attracted by the optimal policy function.[10]

Our first benchmark solution employs a common calibration in stochastic growth models (cf. Posch, 2009b). We use parameters $(\alpha, \theta, \delta, \lambda, \gamma) = (0.5, 2.5, 0.05, 0.2, 0.1)$ and impose the parametric restriction $\rho = ((1-\gamma)^{1-\alpha\theta}-1)\lambda-(1-\alpha\theta)\delta$, which yields an empirically plausible value, $\rho = 0.0178$. Using this calibration the average time between two disasters is $1/\lambda = 5$ years, with each Poisson event destroying 10 percent of the capital stock. Calibrating rare events being less frequent and/or smaller as e.g. for US data (with γ roughly 2.5 percent), our algorithm if anything would improve performance since the true solution is closer to the deterministic guess. As shown in the appendix, in this case consumers choose a constant saving rate $s \equiv 1/\theta$ and the policy function is $C_t = C(K_t) = (1-s)L^{1-\alpha}K_t^{\alpha}$. Thus the optimal jump term is constant, $\tilde{C}(K_t) = (1-\gamma)^{\alpha}$. Although technically a knife-edge solution, the policy functions for solutions around this parameter region are very similar. As shown in Figure 1, the deterministic policy function (for $\lambda = 0$ and/or $\gamma = 0$) and the stochastic policy function differ substantially for our calibration, which illustrates that (potential) rare events can have substantial effects on households' behavior.

Figures 2a and 2b show the absolute and relative error of the numerically obtained policy function compared to the analytical solution, respectively. Both plots indicate that the solution exhibits a high accuracy even for a large deviation from the steady state implied by economically large shocks. The absolute and relative errors compared to the true solution are below 10^{-8} within the most relevant interval between 0 and K^*. The maximum (absolute) errors are below 10^{-5} for values of capital of 150 percent of K^*, which is below the accuracy usually required for economic applications. Economically, this value denotes the error as a fraction of consumption at K_t: with an relative error of 10^{-5}, the consumer is making a \$1 mistake for each \$10,000 spent (Aruoba et al., 2006, p.2499).

Figures 2c and 2d show the absolute and relative change of the policy function, respectively, compared to the previous iteration. It is apparent that both functions are of the same

[9]The literature typically evaluates the performance using Euler equation residuals (see e.g. Judd, 1992). Santos (2000) shows that approximation errors of the policy function are of the same order of magnitude as the Euler equation residuals. Hence, we are able to compare our results with algorithms solving similar models (as in Aruoba, Fernández-Villaverde and Rubio-Ramírez, 2006; Dorofeenko, Lee and Salyer, 2010).

[10]For the backward integration procedure we deviate 10^{-12} in magnitude from the 'steady state' and we choose 10^{-12} as relative error tolerance for the Runge-Kutta procedure (cf. Brunner and Strulik, 2002).

shape and order of magnitude as the numerical errors compared to the analytical solution. This shows that the change of the policy function between two iterations is an excellent approximation for measuring the numerical error of the solution. We make use of this striking similarity to define our criterion function to gauge the accuracy of the numerical solution for the general case where no analytical solution is available.

Our second benchmark solution requires the parametric restriction $\alpha = \theta$, which implies a linear policy function, $C_t = C(K_t) = \phi K_t$.[11] As shown in the appendix, the marginal propensity to consume is $\phi = (\rho - ((1-\gamma)^{1-\theta} - 1)\lambda - (\theta - 1)\delta)/\theta$. Since the policy function is linear, the optimal jump term is constant, $\tilde{C}(K_t) = 1 - \gamma$. For ease of comparison, we choose the same calibration for parameters as above, but a smaller value for the parameter of relative risk aversion (or higher value for the intertemporal elasticity of substitution), $\theta = 0.5$. As shown in Figure 3a both the deterministic policy function and stochastic policy function are indeed linear in the capital stock. Once again, both policy functions differ substantially. Figure 3b shows the optimal jump in consumption with respect to capital, which is again independent of capital.

Figures 4a and 4b show the absolute and relative error of the numerically obtained policy function compared to the analytical solution, respectively. In fact, the solution exhibits a high accuracy of roughly 10^{-15}, close to the machine's precision. Figures 4c and 4d show the absolute and relative change of the policy function, respectively, compared to the previous iteration. Again both measures are of similar shape and order of magnitude.

Our third illustration in Figure 5a shows both the deterministic and the stochastic policy functions for the intermediate case of logarithmic preferences, $\theta = 1$, for which no analytical solution is known. As shown in Figure 5b the optimal jump term now indeed varies with the capital stock and the function $\tilde{C}(K_t)$ is decreasing in capital. As before, we iterate until convergence, i.e., the change of the policy function between two iterations is sufficiently small (cf. Figures 5a and 5b). Because no analytical benchmark solution is available, we now use that both the absolute and relative change of the policy function between two iterations have the same order of magnitude to conclude that the maximum (absolute) error is roughly 10^{-8} within values for capital between 0 and 150 percent of K^*.

Finally, we should emphasize three main points: First, convergence does not depend on parameter restrictions. The algorithm proves to be stable for a wide range of parameters. We restrict the presentation of results to the three calibrations only due to lack of space. Second, computational requirements are rather small. The solution of the model on a standard laptop requires between some seconds and a few minutes. Third, our procedure can be implemented with an average ability in computational skills. While the numerical solution

[11] This solution is well established in macroeconomics (cf. Posch, 2009b, and the references therein).

of the deterministic system is standard, the novel part mainly consists of an interpolation routine based on the Waveform Relaxation idea. However, most software packages provide routines for (spline) interpolation. The Matlab codes and details of our implementation are summarized in a technical appendix, both are available on request.

4.1.2 The economic effects of rare disasters

Asking whether rare disasters lead to higher saving is equivalent to examining whether more uncertainty raises or lowers the marginal propensity to consume. It is well established that the intertemporal substitution effect depresses the marginal propensity to save for risk-avers individuals, as the optimum way to maintain the original utility level when uncertainty increases is to consume more today (and thus avoid facing the disaster risk). In contrast, the income effect is a precautionary savings effect, as higher uncertainty implies a higher probability of low consumption tomorrow against which consumers will protect themselves the more, by consuming less, the more averse they are to intertemporal fluctuations of consumption (cf. Leland, 1968; Sandmo, 1970). By using a nonlinear production technology, the neoclassical theory of growth under uncertainty offers a third channel through which uncertainty has effects on the asymptotic distribution of capital (cf. Merton, 1975).

As shown in Weil (1990), the effect on optimal consumption (or saving) depends on the magnitude of the intertemporal elasticity of substitution, $1/\theta$.[12] Moreover, optimal consumption depends on the degree of curvature of the production technology, α, since the curvature of the policy function matters for effective risk aversion (cf. Posch, 2009a). In case the income effect is relatively small, $\theta < 1$, the presence of rare disasters tends towards higher consumption (cf. Figure 3a). For the case where income and substitution effects balance each other out, $\theta = 1$, the only effect on consumption is due to the concave production technology which depresses the marginal propensity to save (cf. Figure 5a), i.e., the mean capital stock decreases. It is only when the intertemporal elasticity of substitution is small, $\theta > 1$, the precautionary savings motive dominates the substitution effect and eventually the effect of the nonlinear production technology, and savings increase (cf. Figure 1a).

4.2 Lucas' model of endogenous growth with disasters

This section uses the Waveform Relaxation algorithm to solve a stochastic version of the Lucas (1988) endogenous growth model with two controls and two state variables. Motivated by the rare disaster hypothesis, rare events - such as natural disasters - occasionally destroy

[12] Weil (1990) shows that risk aversion, by determining the amplitude of the associated reduction in the certainty equivalent rate of return to saving, only affects the magnitude of the effects described above.

a fraction of the physical capital stock. Our solution method sheds light on the effects on optimal consumption, human capital accumulation, and thus the balanced growth rate.

Specification. Consider a closed economy with competitive markets, with identical agents and a Cobb-Douglas technology, $Y_t = k_t^\alpha (u_t h_t)^{1-\alpha} L$, where $0 < \alpha < 1$. Suppose at date t, L workers (normalized to one) have skill level h_t and own the physical capital stock, k_t. A worker devotes u_t of his non-leisure time to current production, and the remaining $1 - u_t$ to human capital accumulation (improving skills). Hence, the effective aggregate hours devoted to production are $u_t h_t L$. Denoting w_t as the hourly wage rate per unit of effective labor, the individual's labor income at skill h_t is $w_t h_t u_t$. Let the rental rate of physical capital be r_t. For simplicity, there is no capital depreciation, thus k_t evolves according to

$$dk_t = (r_t k_t + w_t u_t h_t - c_t)\, dt - \gamma k_{t-} dN_t, \tag{20}$$

where N_t denotes the number of (natural) disasters up to time t, occasionally destroying $0 < \gamma < 1$ percent of the capital stock k_t at an arrival rate $\lambda \geq 0$.

To complete the model, the research effort $1 - u_t$ devoted to the accumulation of human capital must be linked to h_t. Suppose the technology relating the change of human capital dh_t to the level already attained and the effort devoted to acquiring more is

$$dh_t = (1 - u_t)\vartheta h_t dt. \tag{21}$$

According to (21), if no effort is devoted to human capital accumulation, $u_t = 1$, then non accumulates. If all effort is devoted to this purpose, $u_t = 0$, h_t grows at rate $\vartheta > 0$. In between these extremes, there are no diminishing returns to the stock h_t.

The resource allocation problem faced by the representative individual is to choose a time path for c_t and for u_t in $U_c \subseteq \mathbb{R}_+ \times [0, 1]$ such as to maximize expected life-time utility,

$$\max_{\{c_t, u_t\}_{t=0}^\infty} E_0 \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \quad s.t. \quad (20) \text{ and } (21), \quad (k_0, h_0, N_0) \in \mathbb{R}_+^3, \tag{22}$$

where $\theta > 0$ denotes constant relative risk aversion and ρ is the subjective time preference.

Solution. From the Bellman principle, choosing the controls $c_0, u_0 \in U_c$ requires the *Bellman equation* as a necessary condition for optimality,

$$\begin{aligned}\rho V(k_0, h_0) = \max_{c_0, u_0 \in U_c} \big\{ & c_0^{1-\theta}/(1-\theta) + (r_0 k_0 + w_0 u_0 h_0 - c_0) V_k + (1 - u_0)\vartheta h_0 V_h \\ & + (V((1-\gamma)k_0, h_0) - V(k_0, h_0))\lambda \big\}. \end{aligned} \tag{23}$$

For any $t \in (0, \infty)$, the two *first-order conditions* corresponding to (3) are

$$c_t^{-\theta} - V_k = 0, \tag{24}$$

$$w_t h_t V_k - \vartheta h_t V_h = 0, \tag{25}$$

making the controls a function of the state variables, $c_t = c(k_t, h_t)$ and $u_t = u(k_t, h_t)$.

After some tedious algebra we obtain the Euler equations for consumption and hours. Together with initial and transversality conditions, and budget constraints in (20) and (22), these describe the equilibrium dynamics. We may summarize the reduced form dynamics by defining $\tilde{c}(k_t, h_t) \equiv c((1-\gamma)k_t, h_t)/c(k_t, h_t)$ and $\tilde{u}(k_t, h_t) \equiv u((1-\gamma)k_t, h_t)/u(k_t, h_t)$ as

$$
\begin{aligned}
dk_t &= (r_t k_t + w_t u_t h_t - c_t)\, dt - \gamma k_{t-} dN_t, && \text{(26a)}\\
dh_t &= (1-u_t)\vartheta h_t dt, && \text{(26b)}\\
dc_t &= \frac{r_t - \rho - \lambda + \tilde{c}(k_t, h_t)^{-\theta}(1-\gamma)\lambda}{\theta} c_t dt + (\tilde{c}(k_{t-}, h_{t-}) - 1)c_{t-} dN_t, && \text{(26c)}\\
du_t &= \left(\tfrac{1-\alpha}{\alpha}\vartheta + \left(\tilde{u}(k_t, h_t)^{-\alpha} - (1-\gamma)^{1-\alpha}\right)(1-\gamma)^{\alpha}\lambda \tilde{c}(k_t, h_t)^{-\theta}/\alpha - c_t/k_t + u_t\vartheta\right) u_t dt \\
&\quad + (\tilde{u}(k_t, h_t) - 1)u_{t-} dN_t, && \text{(26d)}
\end{aligned}
$$

and the transversality condition reads (cf. Benhabib and Perli, 1994, p.117)

$$\lim_{t\to\infty} E_0 \left[V_k e^{-\rho t}[k_t - k_t^*] + V_h e^{-\rho t}[h_t - h_t^*]\right] \geq 0$$

for all admissible k_t and h_t, where k_t^* and h_t^* denote the optimal state values.

Balanced growth. From the reduced form system, we can derive the balanced growth rate of physical capital, human capital and consumption of the conditional deterministic system (conditioned on no disasters) as follows. First, we can neglect the stochastic integrals because for the case with no disasters, $dN_t \equiv 0$. Second, similar to the deterministic model, the condition optimal research effort is constant, such that $du_t = 0$ must hold.

Now, for $du_t = 0$ research effort along the balanced growth path is implicitly given by $-\vartheta u^* = \frac{1-\alpha}{\alpha}\vartheta + (\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha})(1-\gamma)^{\alpha}\lambda\tilde{c}^{-\theta}/\alpha - c/k$, where $\tilde{u} \equiv \tilde{u}(k_t, h_t)$, $\tilde{c} \equiv \tilde{c}(k_t, h_t)$ and $c/k \equiv c_t/k_t$ are constants. This property of the jump terms implies that asymptotically, $\tilde{c}(k_t, h_t) = \tilde{c}(k_t/h_t)$. Similarly, along this balanced growth path the other equations imply

$$g^k = r^*/\alpha - c/k, \quad g^h = (1-u^*)\vartheta, \quad g^c = \left(r^* - \rho - \lambda + \tilde{c}^{-\theta}(1-\gamma)\lambda\right)/\theta.$$

Since c_t/k_t is constant, c_t and k_t must grow at the same rate,

$$g^k = g^c \quad \Rightarrow \quad c/k = \left(r^* - \rho - \lambda + \tilde{c}^{-\theta}(1-\gamma)\lambda\right)/\theta + r^*/\alpha.$$

Along this path r^* is constant, which requires that k_t and h_t must growth at the same rate,

$$
\begin{aligned}
g^k = g^h \quad \Rightarrow \quad r^*/\alpha - c/k &= (1-u^*)\vartheta \\
\Leftrightarrow \ r^* &= \vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)\tilde{c}^{-\theta}(1-\gamma)^{\alpha}\lambda.
\end{aligned}
$$

Hence, the balanced growth rate of the *conditional* deterministic system is

$$g \equiv \left(\vartheta - \rho - \lambda + (1-\gamma)^{\alpha}\tilde{u}^{-\alpha}\tilde{c}^{-\theta}\lambda\right)/\theta, \tag{27}$$

which implies

$$
\begin{aligned}
c/k \;=\; & \big(\vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)\tilde{c}^{-\theta}(1-\gamma)^{\alpha}\lambda - \rho - \lambda + \tilde{c}^{-\theta}(1-\gamma)\lambda\big)/\theta \\
& + \big(\vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)\tilde{c}^{-\theta}(1-\gamma)^{\alpha}\lambda\big)/\alpha.
\end{aligned}
$$

The growing variables of the reduced-form system c_t, h_t, and k_t in (26) need to be scaled such that they approach some stationary steady-state values (scale-adjustment).

Scale-adjusted dynamics. In what follows, we simply subtract the endogenous balanced growth rate (27) from the reduced-form system in instantaneous growth rates to obtain scale-adjusted variables. The scale-adjusted system (conditioned on no disasters) reads

$$
\begin{aligned}
d\ln k_t &= \left(r_t + w_t u_t h_t/k_t - c_t/k_t - g\right)dt, && \text{(28a)}\\
d\ln h_t &= (\vartheta - u_t\vartheta - g)dt, && \text{(28b)}\\
d\ln c_t &= \left(\left(r_t - \rho - \lambda + \tilde{c}(k_t,h_t)^{-\theta}(1-\gamma)\lambda\right)/\theta - g\right)dt, && \text{(28c)}\\
du_t &= \left(\tfrac{1-\alpha}{\alpha}\vartheta + \left(\tilde{u}(k_t,h_t)^{-\alpha} - (1-\gamma)^{1-\alpha}\right)(1-\gamma)^{\alpha}\lambda\tilde{c}(k_t,h_t)^{-\theta}/\alpha - c_t/k_t\right)u_t dt \\
&\quad + \vartheta u_t^2 dt, && \text{(28d)}
\end{aligned}
$$

where g follows iteratively from (27).

Note that in general it is not possible to compute the steady state levels in terms of variables k^*, h^*, c^*, and u^* from system (28). We presume that the stochastic model inherits this characteristic from its deterministic counterpart, which exhibits a ray of steady states, i.e., a center manifold of stationary equilibria (cf. Lucas, 1988; Caballé and Santos, 1993). Each point on this ray differs with respect to the level of physical and human capital and, hence, consumption the economy can generate. The particular stationary equilibrium, to which the economy finally converges is determined by the initial values of physical and human capital. Since in general the functions $\tilde{c}$ and $\tilde{u}$ are not known for the stochastic counterpart of the model, we are not able to prove this property for the general case. However, for a specific parametric restriction we obtain a closed-form solution and indeed provide a proof of this property below. Moreover, our numerical results confirm that the stochastic model indeed exhibits a ray of steady states. A 'steady-state' value in the stochastic setup again refers to the value the economy converges if no disasters occur.

We are now prepared to solve this (scale-adjusted) system using the Relaxation algorithm together with the Waveform relaxation idea.

4.2.1 Evaluation of the algorithm

We calculate numerical solutions for the Lucas model employing a benchmark calibration, for which an analytical solution is available. Again, we compare the numerical and analytical solutions to evaluate the algorithm's accuracy. Moreover, we calculate numerical solutions for a second calibration, for which no analytical solution is available.

Because this model has two state variables, we choose the Relaxation algorithm to solve system (12) (cf. Trimborn et al., 2008). As already mentioned this algorithm is capable of solving deterministic systems with multiple state variables. Moreover, the algorithm can also solve models that exhibit a center manifold of stationary equilibria. Since the method calculates the solution path as a whole, the particular conditional steady state to which the economy converges is determined numerically.

Our benchmark solution uses the calibration $(\alpha, \vartheta, \lambda, \gamma, \rho) = (0.75, 0.075, 0.2, 0.1, 0.03)$ and the parametric restriction $\theta = \alpha$. As shown in the appendix, in this case consumers optimally choose constant hours, $u_t = u = (\rho - (1-\theta)\vartheta)/(\alpha\vartheta)$ and optimal consumption does not depend on human capital and is linear in physical capital, $c_t = c(k_t, h_t) = \varphi k_t$. $\varphi = (\rho - ((1-\gamma)^{1-\theta} - 1)\lambda)/\theta$ denotes the marginal propensity to consume with respect to physical capital. Since the policy function is linear in physical capital the optimal jump terms are constant, $\tilde{c}(k_t, h_t) = 1 - \gamma$ and trivially $\tilde{u}(k_t, h_t) = 1$. Observe that this solution is very similar to the neoclassical growth model, though the growth rate is endogenous. From (27) we find that for $\alpha = \theta$, the balanced growth rate (in normal times, after the transition) is not affected by the presence of rare events, $g = (\vartheta - \rho)/\theta$. Below we compare our numerical solution obtained by the Waveform Relaxation algorithm with the analytical solution.

Figures 7a and 7b, respectively, show the optimal level of consumption and the optimal jump in consumption with respect to physical capital and human capital. Note that the optimal jump in consumption is independent of both physical capital and human capital. Similar to the neoclassical growth model, we find that the deterministic policy function for consumption (for $\lambda = 0$ and/or $\gamma = 0$) and the stochastic counterpart differ substantially. Moreover, the center manifold of stationary equilibria of (scale-adjusted) values for human capital and physical capital is different from the deterministic model, but as discussed above this property is inherited by the stochastic model from its deterministic counterpart.

Figures 8a and 8b show the absolute and relative error of consumption for the computed mesh grid of physical and human capital. Given the nature of the problem, the (absolute) errors are extremely small, not exceeding 10^{-8} in magnitude. As explained above, this level of accuracy is higher than what is usually required for most economic applications. Figures 8c and 8d show the absolute and relative change in the policy function for consumption, respectively, compared to the previous iteration. It is apparent that both functions are of the same shape and order of magnitude as the numerical errors compared to the analytical solution, which helps to gauge the numerical error of the solution in the general case.

Similarly to the case of consumption, Figures 9a and 9b show the optimal level of hours worked and the optimal jump with respect to physical capital and human capital. Hours are independent of capital goods along the transition and, hence, do not adjust in case of a Poisson jump. Figures 10a and 10b show the absolute and relative error of hours worked, whereas the absolute and relative change in the policy function for hours compared to the previous iteration are shown in Figures 10c and 10d, respectively. Again, the maximum (absolute) errors are very small and do not exceed 10^{-6}.

As an illustration for a case where closed-form solutions are not available, we compare our benchmark solution to the case of logarithmic preferences, $\theta = 1$. Figures 11 and 13 show the optimal policy functions for consumption and hours and the optimal jump in consumption and hours, respectively. We find that the optimal levels and their jump terms now depend on the level of physical capital and human capital. While the level of optimal consumption is increasing in both capital goods, hours are increasing in human capital but decreasing in physical capital. Hence, countries with an abundant supply of human capital but scarce supply of physical capital tend to supply the most hours to production.

Again we would like to emphasize that we are able to calculate policy functions not only for the parametric restrictions presented above, but for a wider range of parameter values. However, the algorithm is not as stable as for the one-dimensional case and is less precise mainly due to interpolation problems. Eventually, for extreme combination of parameter values problems of convergence might occur, or at least the procedure needs refinement with respect to the chosen mesh and/or interpolation method. Since our main objective is to show that multiple state variables do not pose conceptional problems for our solution method, we leave this work for future research. The Matlab codes and details of our implementation are summarized in a technical appendix, both available on request.

4.2.2 The economic effects of rare disasters

The Lucas model of endogenous growth has several channels through which uncertainty enters in the economic decisions, and thus optimal plans will be affected when consumers face more uncertainty. First, uncertainty will affect the consumption/saving decision as in the neoclassical growth model. Second, uncertainty will enter the optimal allocation problem of hours devoted to production and human capital accumulation. Finally, their optimal behavior takes account of the effect on the (conditional) balanced growth path.

As shown in Figures 7a and 11a, the level of (scale-adjusted) consumption increases for both calibrations, thus the dominating channel is the intertemporal substitution effect, i.e., to consume more today (and thus avoid facing the disaster risk). In other words, the intertemporal elasticity of substitution is sufficiently elastic to compensate the precautionary savings effect. This is in line with the result from the neoclassical growth model.

But consumption is no longer the only way to accommodate the presence of risk. From Figure 13a, for the case of logarithmic preferences with $\theta = 1$, we find that optimal hours decrease due to the presence of rare disasters (a level shift). Intuitively, consumers prefer to invest more in human capital accumulation which - in contrast to the physical capital good - is not subject to disaster risk. Though it seems an intuitive response from an asset pricing perspective, we find that this result cannot be generalized. As from Figure 9a, optimal hours are independent of the disaster risk. Supplying less hours for production also has an income effect, which in the case of $\alpha = \theta$ exactly offsets the previous effect. This example illustrates that it is important to study the effects of uncertainty within a dynamic stochastic general equilibrium (DSGE) model, in order to avoid missing potentially important feedback mechanisms when focusing on partial equilibrium effects only.

As from (27), the balanced growth rate (in normal times, after the transition) depends both on the optimal jump terms for consumption and for hours. In our numerical solution for $\theta = 1$, the balanced growth rate of the deterministic system ($\lambda = 0$ and/or $\gamma = 0$), which gives $(\vartheta - \rho)/\theta = 4.5\%$ increases by roughly 0.2 percentage points to $g = 4.7\%$ due to the presence of rare disasters (which happen not to occur in normal times). An intuitive explanation of this effect is indeed the shift of optimal hours supplied to human capital accumulation, and thus implying a higher growth rate in times without disasters.

5 Conclusion

In this paper we propose a simple and powerful method for determining the transitional dynamics in continuous-time DSGE models under Poisson uncertainty. Our contribution is to show how existing algorithms can be extended with an additional layer when we allow for the possibility of rare events in the form of Poisson uncertainty.

We illustrate the algorithm by computing the stochastic neoclassical growth model and a stochastic version of the Lucas model motivated by the Barro-Rietz rare disaster hypothesis. As a novelty in dynamic stochastic general equilibrium (DSGE) models, analytical solutions serve as a benchmark in order to address the numerical accuracy. These analytical solutions are available for plausible parametric restrictions. We find that even for non-linear policy functions, the numerical error is extremely small.

From an economic perspective, we show that the simple awareness of the possibility of infrequent large economic shocks affects optimal decisions and thus economic growth. The effect is economically important and thus needs to be explored in future research.

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A Appendix

A.1 A closed-form solution to the Ramsey model

The idea is to provide an educated guess of the value function and then derive conditions under which it satisfies both, the first-order condition and the maximized Bellman equation.

Suppose that

$$V(K_t) = \frac{\mathbb{C}_1 K_t^{1-\alpha\theta}}{1-\alpha\theta}. \tag{29}$$

From (17), optimal consumption per effective worker is a constant fraction of income,

$$C_t^{-\theta} = \mathbb{C}_1 K_t^{-\alpha\theta} \quad \Leftrightarrow \quad C_t = C(K_t) = \mathbb{C}_1^{-1/\theta} K_t^{\alpha}.$$

Now use the maximized Bellman equation together with CRRA utility $u(C_t) = C_t^{1-\theta}/(1-\theta)$ and insert the solution candidate,

$$\rho V(K_t) \;=\; \frac{C(K_t)^{1-\theta}}{1-\theta} + (K_t^{\alpha} L^{1-\alpha} - C(K_t) - \delta K_t) V_K + (V((1-\gamma)K_t) - V(K_t))\,\lambda$$

which is equivalent to

$$\begin{aligned}
(\rho+\lambda)\frac{\mathbb{C}_1 K_t^{1-\alpha\theta}}{1-\alpha\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} K_t^{\alpha-\alpha\theta}}{1-\theta} + \left(K_t^{\alpha} L^{1-\alpha} - \mathbb{C}_1^{-1/\theta} K_t^{\alpha} - \delta K_t\right) \mathbb{C}_1 K_t^{-\alpha\theta} \\
&\quad + \frac{\mathbb{C}_1 K_t^{1-\alpha\theta}}{1-\alpha\theta}(1-\gamma)^{1-\alpha\theta}\lambda \\
\Leftrightarrow 0 &= \frac{\theta}{1-\theta}\mathbb{C}_1^{-\frac{1}{\theta}} + L^{1-\alpha} - (\rho + (1-\alpha\theta)\delta + \lambda - (1-\gamma)^{1-\alpha\theta}\lambda)\frac{K_t^{1-\alpha}}{1-\alpha\theta}
\end{aligned}$$

which has a solution for $\mathbb{C}_1^{-1/\theta} = (\theta-1)/\theta L^{1-\alpha}$ and

$$\rho = (1-\gamma)^{1-\alpha\theta}\lambda - \lambda - (1-\alpha\theta)\delta. \tag{30}$$

For reasonable parametric calibrations equation (30) is satisfied. Though being a special case, a Keynesian consumption function could be an admissible policy function for the neoclassical model (cf. also Chang, 1988). Its plausibility is an empirical question.

A.2 A closed-form solution to the Lucas model

We start with an educated guess on the value function and then derive conditions under which it actually is the unique solution of the optimal stochastic control problem.

Suppose that

$$V(k_t, h_t) = \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta}. \tag{31}$$

From (24), we obtain that optimal consumption is a linear function in the capital stock

$$c_t^{-\theta} = \mathbb{C}_1 k_t^{-\theta} \quad \Rightarrow \quad c(k_t, h_t) = \mathbb{C}_1^{-\frac{1}{\theta}} k_t \tag{32}$$

Similarly, from (25) we obtain the optimal share of hours allocated to production, u_t,

$$\begin{aligned} w_t h_t \mathbb{C}_1 k_t^{-\theta} &= \vartheta h_t \mathbb{C}_2 h_t^{-\theta} \\ \Leftrightarrow \quad (1-\alpha) k_t^{\alpha} (u_t h_t)^{-\alpha} \mathbb{C}_1 k_t^{-\theta} &= \vartheta \mathbb{C}_2 h_t^{-\theta} \\ \Leftrightarrow \quad u(k_t, h_t) &= \left(\frac{\vartheta}{(1-\alpha)} \frac{\mathbb{C}_2}{\mathbb{C}_1} h_t^{\alpha-\theta} k_t^{\theta-\alpha} \right)^{-\frac{1}{\alpha}} \end{aligned}$$

where we used that $w_t = (1-\alpha) k_t^{\alpha} (u_t h_t)^{-\alpha}$. Observe that for the parametric restriction $\alpha = \theta$, optimal hours allocated to production becomes a constant,

$$\alpha = \theta \quad \Rightarrow \quad u(k_t, h_t) = \left(\frac{\vartheta}{(1-\alpha)} \frac{\mathbb{C}_2}{\mathbb{C}_1} \right)^{-\frac{1}{\alpha}}.$$

Using the maximized Bellman equation, we may write with $r_t = \alpha k_t^{\alpha-1} (u_t h_t)^{1-\alpha}$

$$\begin{aligned} \rho V(k_t, h_t) &= \frac{c(k_t, h_t)^{1-\theta}}{1-\theta} + (r_t k_t + w_t u(k_t, h_t) h_t - c(k_t, h_t)) V_k + (1 - u(k_t, h_t)) \vartheta h_t V_h \\ &\quad + (V((1-\gamma) k_t, h_t) - V(k_t, h_t)) \lambda \\ &= \frac{c(k_t, h_t)^{1-\theta}}{1-\theta} + (k_t^{\alpha} (u(k_t, h_t) h_t)^{1-\alpha} - c(k_t, h_t)) V_k + (1 - u(k_t, h_t)) \vartheta h_t V_h \\ &\quad + (V((1-\gamma) k_t, h_t) - V(k_t, h_t)) \lambda. \end{aligned}$$

Inserting the guess for the value function gives

$$\begin{aligned} (\rho + \lambda) \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} &= \frac{c(k_t, h_t)^{1-\theta}}{1-\theta} + (k_t^{\alpha} (u(k_t, h_t) h_t)^{1-\alpha} - c(k_t, h_t)) \mathbb{C}_1 k_t^{-\theta} \\ &\quad + (1 - u(k_t, h_t)) \vartheta h_t \mathbb{C}_2 h_t^{-\theta} + \frac{\mathbb{C}_1 (1-\gamma)^{1-\theta} k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} \lambda \end{aligned}$$

Now insert the policy function for consumption $c(k_t, h_t)$,

$$\begin{aligned} (\rho + \lambda) \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} k_t^{1-\theta}}{1-\theta} + \left(k_t^{\alpha} (u(k_t, h_t) h_t)^{1-\alpha} - \mathbb{C}_1^{-\frac{1}{\theta}} k_t \right) \mathbb{C}_1 k_t^{-\theta} \\ &\quad + (1 - u(k_t, h_t)) \vartheta \mathbb{C}_2 h_t^{1-\theta} + \frac{\mathbb{C}_1 (1-\gamma)^{1-\theta} k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} \lambda. \end{aligned}$$

and employ the restriction $\theta = \alpha$, where optimal hours is constant, $u(k_t, h_t) = u$, it reads

$$\begin{aligned}(\rho+\lambda)\frac{\mathbb{C}_1 k_t^{1-\theta}+\mathbb{C}_2 h_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} k_t^{1-\theta}}{1-\theta} + \left(u^{1-\alpha}h_t^{1-\alpha} - \mathbb{C}_1^{-\frac{1}{\theta}} k_t^{1-\theta}\right)\mathbb{C}_1 \\ &\quad +(1-u)\vartheta\mathbb{C}_2 h_t^{1-\theta} + \frac{\mathbb{C}_1(1-\gamma)^{1-\theta}k_t^{1-\theta}+\mathbb{C}_2 h_t^{1-\theta}}{1-\theta}\lambda.\end{aligned}$$

Collecting terms, we obtain

$$\begin{gathered}\left(\rho+\lambda-\theta\mathbb{C}_1^{-\frac{1}{\theta}}-(1-\gamma)^{1-\theta}\lambda\right)\mathbb{C}_1 k_t^{1-\theta} = \\ \left((1-\theta)u^{1-\alpha}\mathbb{C}_1+(1-\theta)(1-u)\vartheta\mathbb{C}_2-(\rho+\lambda)\mathbb{C}_2+\mathbb{C}_2\lambda\right) h_t^{1-\theta}.\end{gathered}$$

Hence, the first constant is pinned down by $\mathbb{C}_1 = (\theta/(\rho+\lambda-(1-\gamma)^{1-\theta}\lambda))^{\theta}$. Inserting u, finally pins down the second constant,

$$\begin{aligned}\rho\mathbb{C}_2 &= (1-\theta)u^{1-\alpha}\mathbb{C}_1+(1-\theta)(1-u)\vartheta\mathbb{C}_2 \\ \Leftrightarrow\ \frac{\rho}{(1-\theta)\vartheta} &= \frac{1}{1-\alpha}\left(\frac{\vartheta}{(1-\alpha)}\right)^{-\frac{1}{\alpha}}\mathbb{C}_1^{\frac{1}{\alpha}}\mathbb{C}_2^{-\frac{1}{\alpha}}+1-\left(\frac{\vartheta}{(1-\alpha)}\right)^{-\frac{1}{\alpha}}\mathbb{C}_1^{\frac{1}{\alpha}}\mathbb{C}_2^{-\frac{1}{\alpha}} \\ \Leftrightarrow\ \frac{\rho-(1-\theta)\vartheta}{(1-\theta)\vartheta} &= \frac{\alpha}{1-\alpha}\left(\frac{\vartheta}{(1-\alpha)}\right)^{-\frac{1}{\alpha}}\mathbb{C}_1^{\frac{1}{\alpha}}\mathbb{C}_2^{-\frac{1}{\alpha}} \\ \Leftrightarrow\ \mathbb{C}_2^{\frac{1}{\alpha}} &= \frac{\alpha}{1-\alpha}\left(\frac{\vartheta}{(1-\alpha)}\right)^{-\frac{1}{\alpha}}\frac{(1-\theta)\vartheta}{\rho-(1-\theta)\vartheta}\mathbb{C}_1^{\frac{1}{\alpha}} \\ \Rightarrow\quad \mathbb{C}_2 &= \left(\frac{\alpha\vartheta}{\rho-(1-\theta)\vartheta}\right)^{\alpha}\frac{1-\alpha}{\vartheta}\left(\frac{\theta}{\rho+\lambda-(1-\gamma)^{1-\theta}\lambda}\right)^{\theta}\end{aligned}$$

Observe that we solved *not only* for some balanced growth path, but for the whole transition path for a parameter restriction.

To summarize, for $\alpha = \theta$ we obtain

$$c(k_t,h_t) = c(k_t) = \frac{\rho+\lambda-(1-\gamma)^{1-\theta}\lambda}{\theta}k_t \tag{33}$$

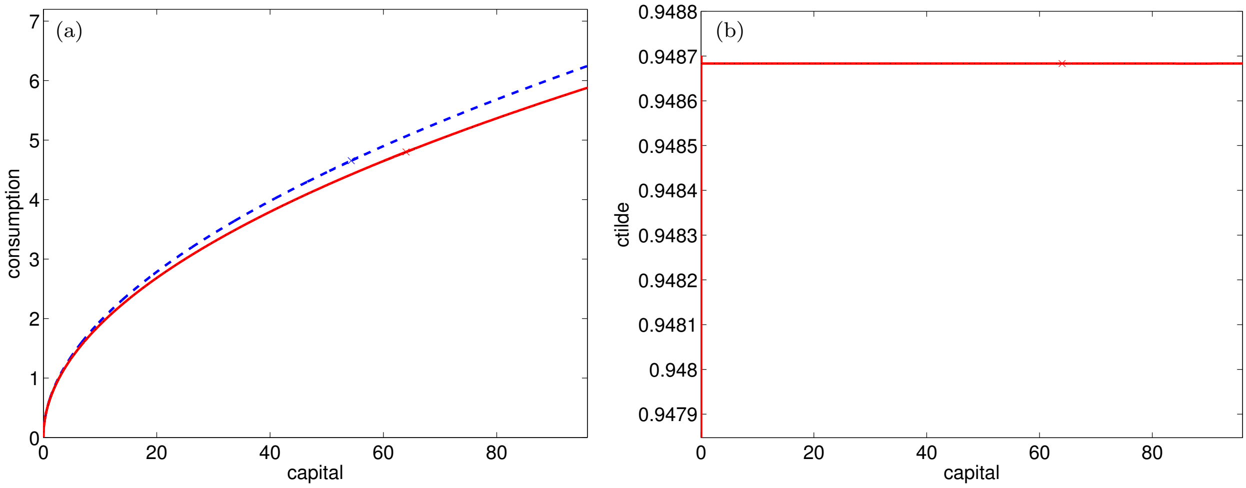
$$u(k_t,h_t) = u = \frac{\rho-(1-\theta)\vartheta}{\alpha\vartheta} \tag{34}$$

Hence, individuals relatively prefer more consumption (or less investment) but work the same hours compared to the traditional model for $\alpha = \theta$. Note that this analytical solution to the stochastic extension of the Lucas model is novel.

B Figures

B.1 A neoclassical growth model with disasters

Figure 1: Policy functions and optimal jump in the neoclassical growth model (1)



Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the neoclassical growth model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of capital for the calibration $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 2.5, 0.05, 0.2, 0.1, 0.0178)$, which implies a constant saving rate.

Figure 2: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration

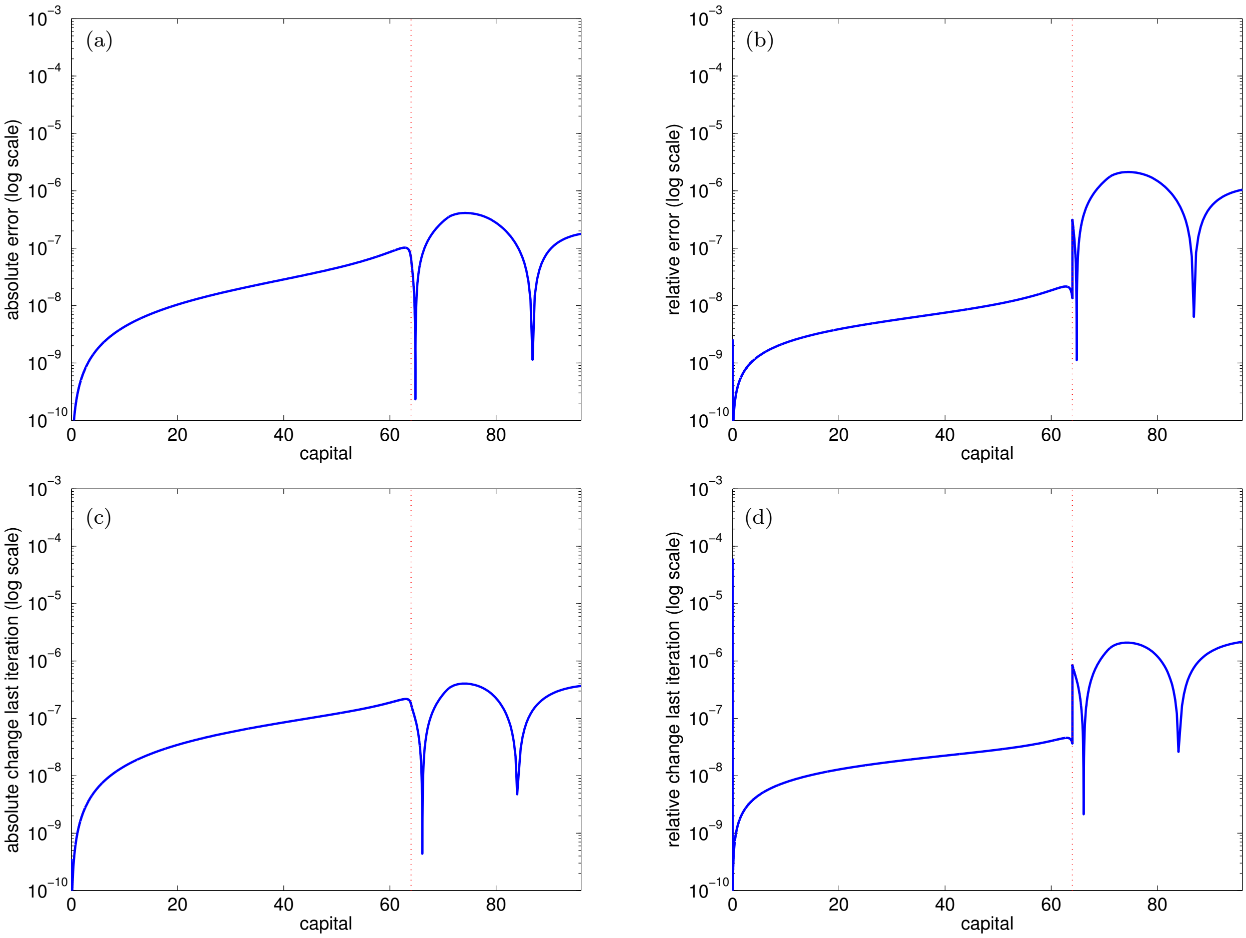
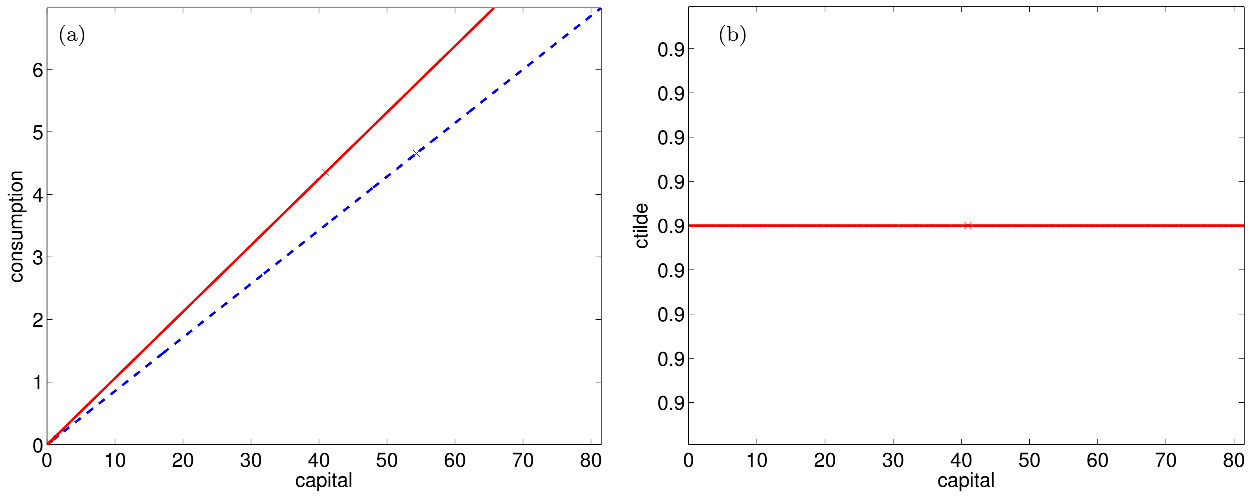


Figure 3: Policy functions and optimal jump in the neoclassical growth model (2)



Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the neoclassical growth model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of capital for the calibration $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 0.5, 0.05, 0.2, 0.1, 0.0178)$, which implies a linear policy function.

Figure 4: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration

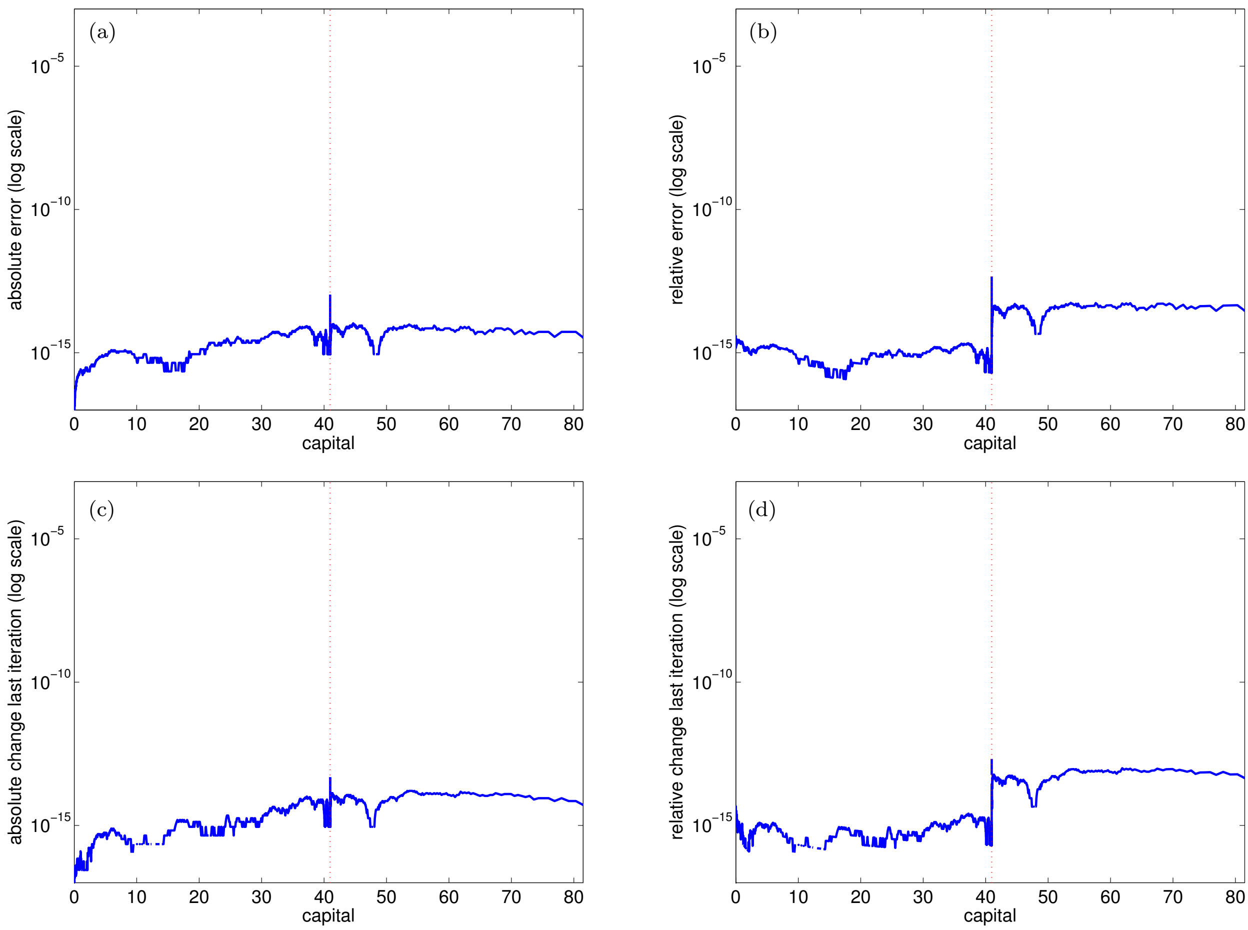
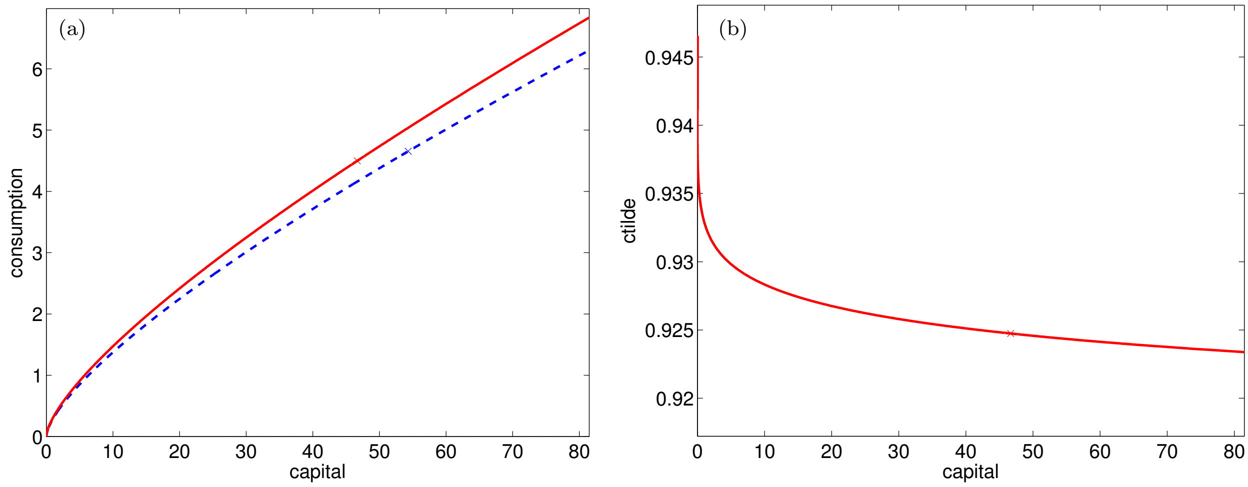
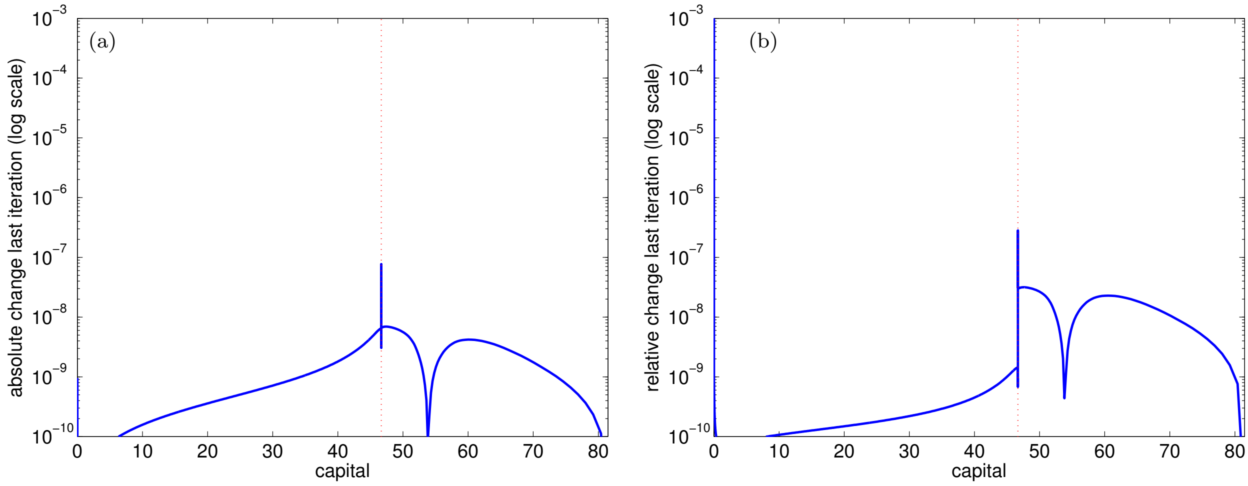


Figure 5: Policy functions and optimal jump in the neoclassical growth model (3)



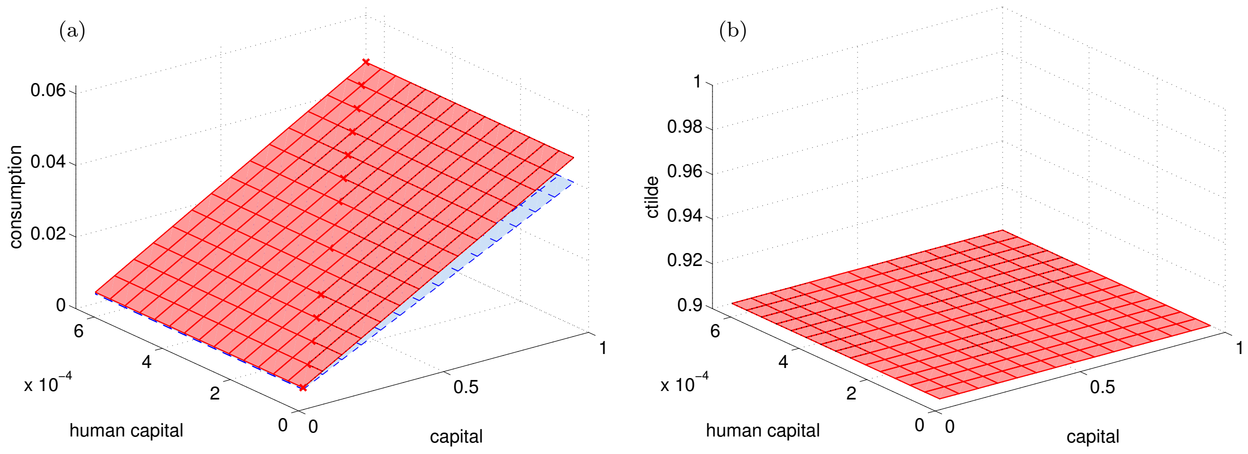
Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the neoclassical growth model (no analytical benchmark solution available), and (b) the optimal jump as a function of capital for the calibration $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 1, 0.05, 0.2, 0.1, 0.0178)$.

Figure 6: Absolute and relative error compared to the policy function of the last iteration (no analytical errors available)



B.2 Lucas' model of endogenous growth with disasters

Figure 7: Policy functions and optimal jump for consumption in the Lucas model (1)



Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the Lucas model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 0.75, 0.075, 0.2, 0.1, 0.03)$, which implies a linear policy plane.

Figure 8: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration

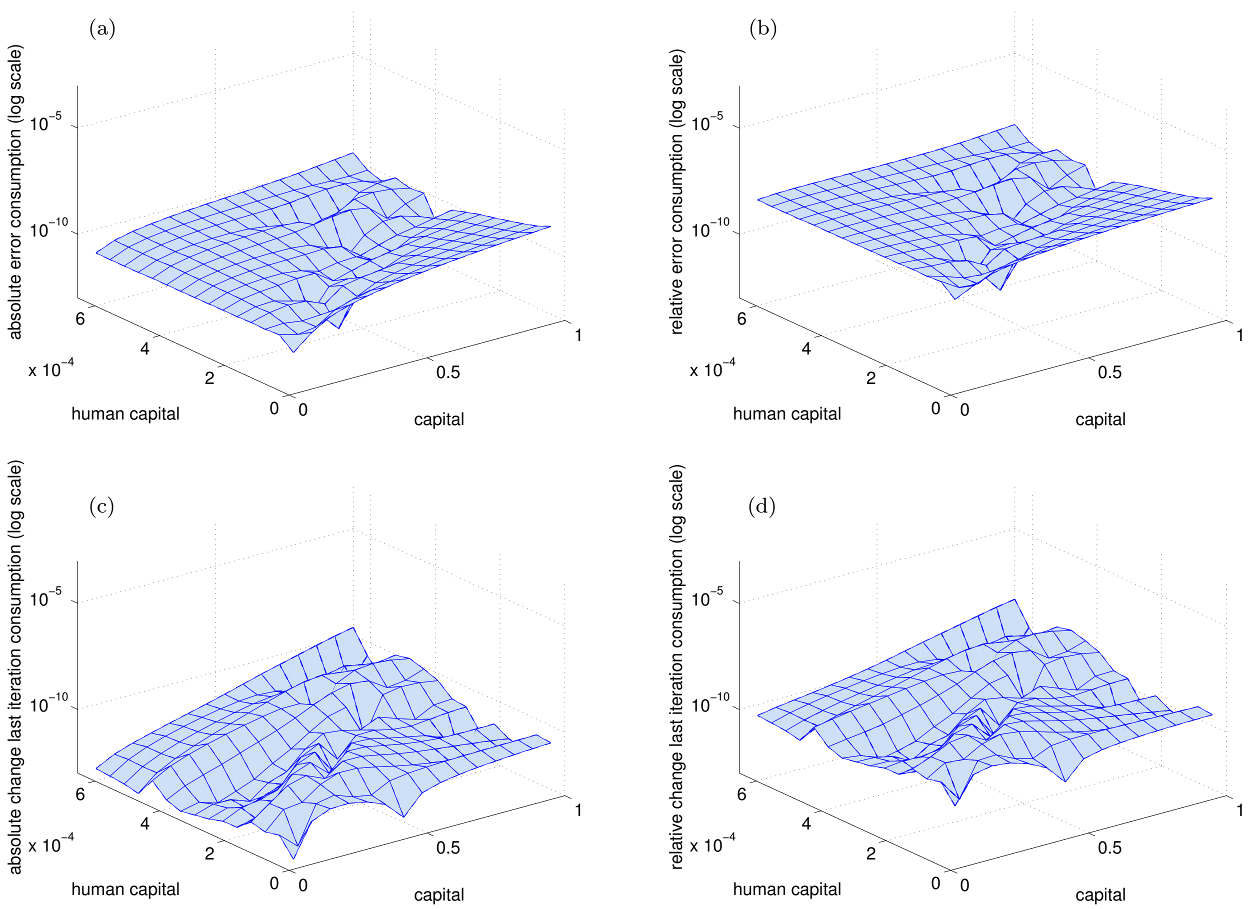
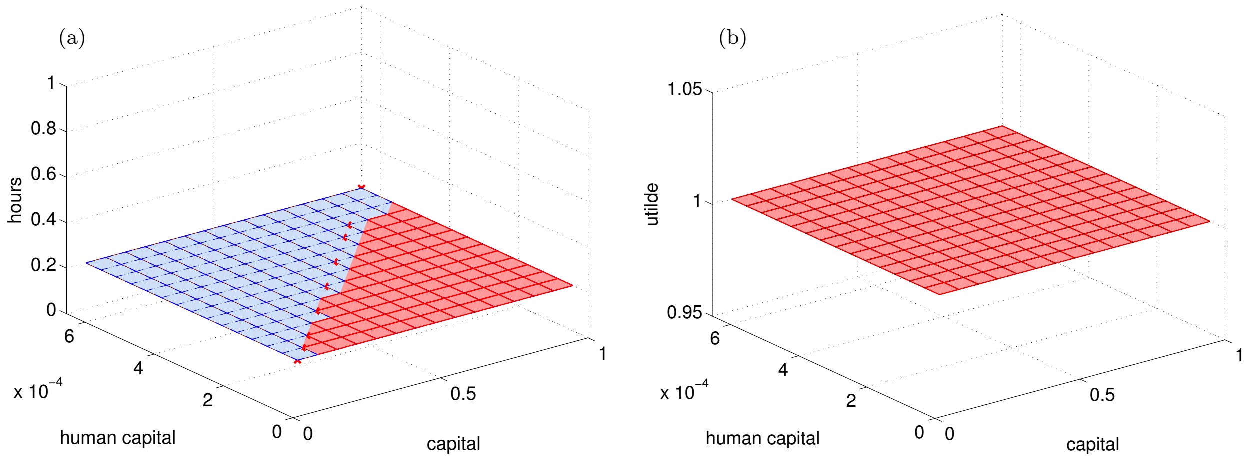


Figure 9: Policy functions and optimal jump for hours in the Lucas model (1)



Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the Lucas model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 0.75, 0.075, 0.2, 0.1, 0.03)$, which implies a linear policy plane.

Figure 10: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration

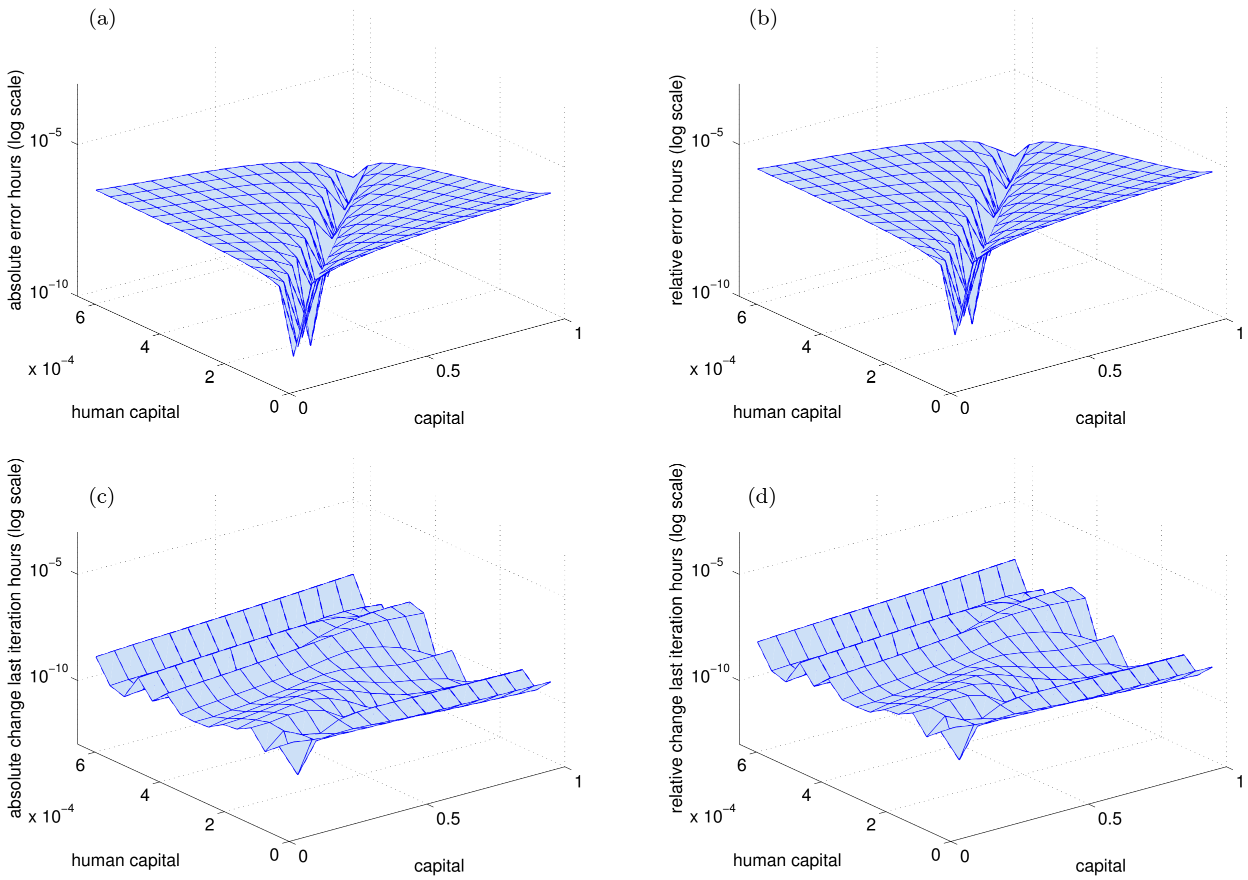
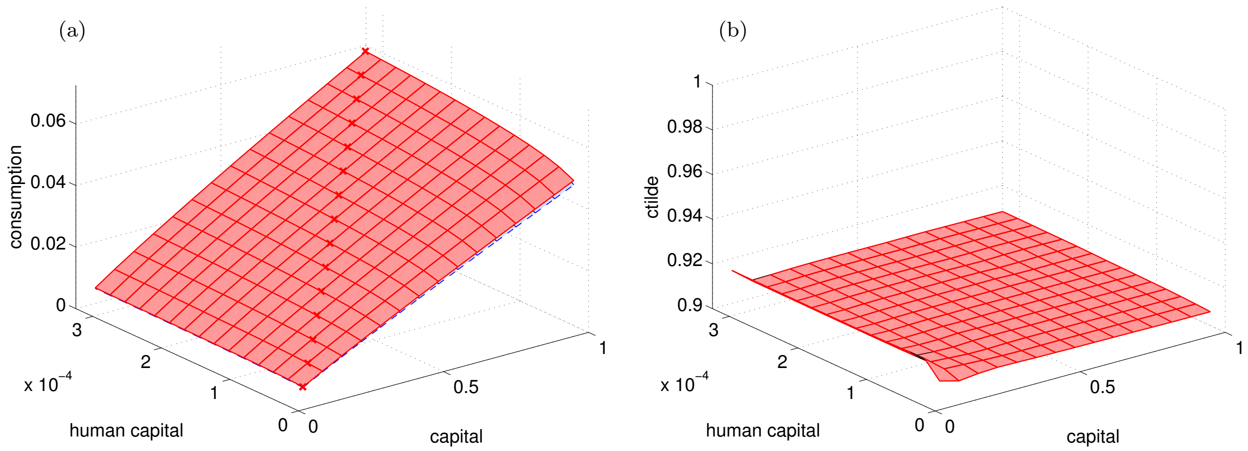


Figure 11: Policy functions and optimal jump for consumption in the Lucas model (2)



Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the Lucas model (no analytical benchmark solution available), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 1, 0.075, 0.2, 0.1, 0.03)$.

Figure 12: Absolute and relative error compared to the policy function of the last iteration (no analytical errors available)

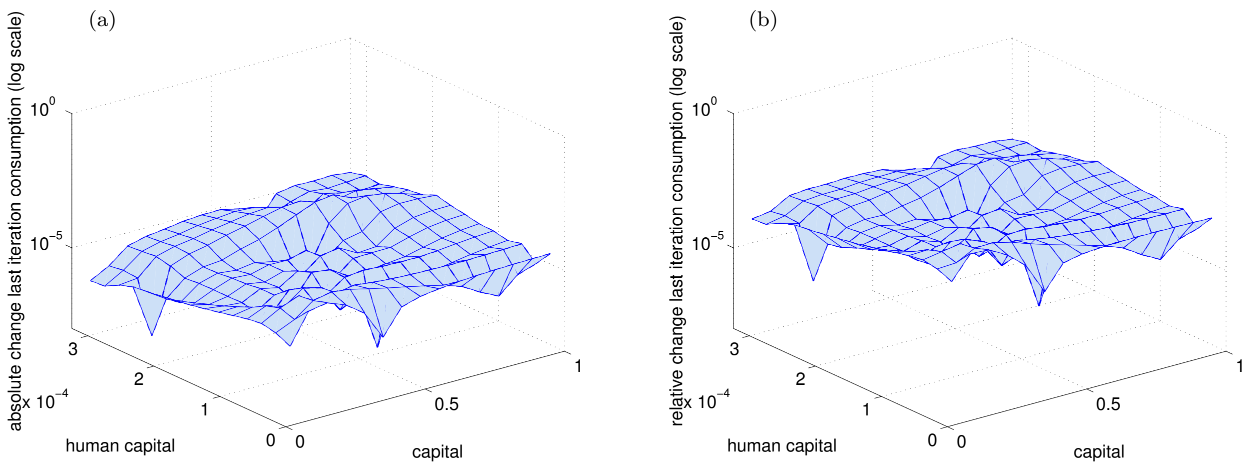
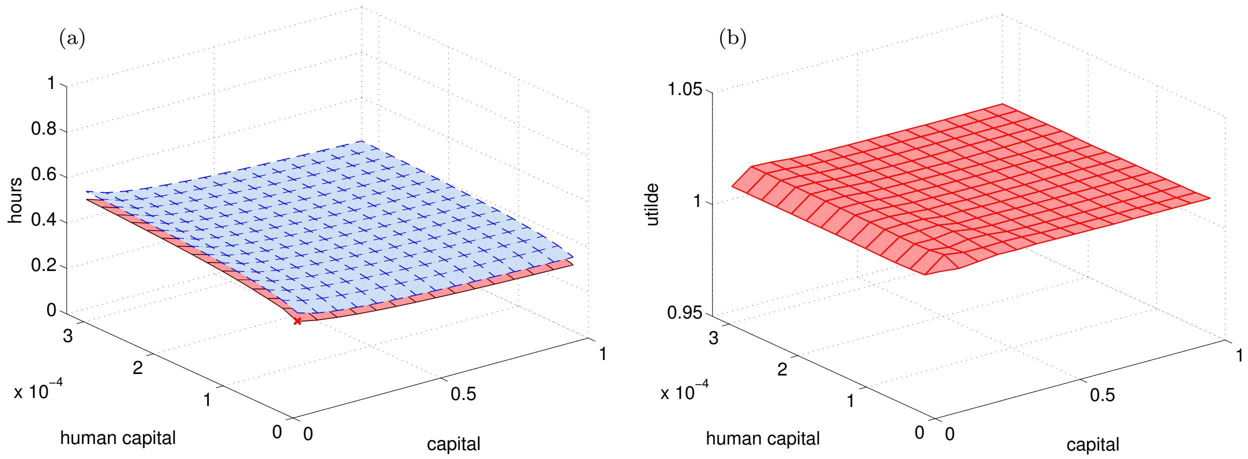


Figure 13: Policy functions and optimal jump for hours in the Lucas model (2)



Notes: These figures show (a) the optimal policy functions deterministic (dashed) vs. stochastic (solid) in the Lucas model (no analytical benchmark solution available), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 1, 0.075, 0.2, 0.1, 0.03)$.

Figure 14: Absolute and relative error compared to the policy function of the last iteration (no analytical errors available)