# Resurrecting the New Keynesian Model: (Un)conventional Policy and the Taylor rule

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**Abstract**

This paper explores the ability of the New Keynesian (NK) model to explain the recent periods of quiet and stable inflation at near-zero nominal interest rates. We show that temporary and permanent shocks to the natural rate (and inflation) are sufficient for the ability of the simple NK model to explain the recent facts. Based on the identified shocks from a novel approach, we show that the model can replicate key macroeconomic variables in accordance with the term structure of interest rates. We find that the term structure helps to identify permanent shocks. Our analysis is restricted to an active role of monetary policy and the traditional regions of (local) determinacy. We also show that capturing highly nonlinear dynamics can be useful to generate a prolonged period of near-zero interest rates as a policy choice.

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# 1. Introduction

> "Theories ultimately rise and fall on their ability to organize and interpret facts." (Cochrane, 2011, p.566)

In the aftermath of the financial crisis, New Keynesian (NK) theory has fallen on hard times. Once being a pillar of macroeconomics, in particular monetary economics, it has been criticized on both the theoretical and the empirical ends. Consider the simplest three-equation NK model with rational expectations and active monetary policy, and the cut in interest rates from 5.25% in 2007 to 0.25% by the end of 2008 (cf. Figure 1). When interpreting this cut as an exogenous but transitory monetary policy shock, the NK model predicts a counterfactual rise of inflation to more than 4 percent.[1] A more plausible scenario is that the fall in interest rates is a *response* to other shocks, usually affecting the natural rate. However, the subsequent episode of an apparently binding zero-lower bound (ZLB), referred to as the zero-interest-rate policy (ZIRP) period, even intensifies the criticism. If the economy entered a liquidity-trap scenario, the NK model would predict a deep recession with deflation (cf. Werning, 2012; Cochrane, 2017b). But nothing happened. If anything, core inflation (excluding food and energy) declined moderately to values around 1 percent in 2010. So what happened? Is the Taylor principle applicable in a world where interest rates stopped moving more than one-for-one with inflation? Cochrane (2017a) shows that alternative doctrines, including old-Keynesian models and the monetarist view, fail in explaining the ZIRP period, when the Fed drastically decreased interest rates and embarked on immense (unconventional) open market operations.

So the open question is on the ability of the NK model to organize our thoughts and interpret the recent facts. Can we reconcile the dynamics of key macroeconomic variables and the term structure of interest rates with the model predictions? Del Negro, Giannoni, and Schorfheide (2015) challenge the criticism by showing that a medium-sized NK model of Smets and Wouters (2007), based on Christiano, Eichenbaum, and Evans (2005), with time-varying inflation target and financial frictions is able to predict a sharp decline in output without forecasting a large drop in inflation. Based on this finding, do we have to abandon the three-equation NK model? In this paper, we allow for both temporary and permanent shocks to inflation and the interest rate. Based on this tweak, we show that the simple NK model can be used to interpret the data. Central to the identification of temporary and permanent shocks is the ability to replicate the yield curve, which from the expectation hypothesis relates to the perceptions about future interest rates and inflation. We also shed light on potential primitive sources of permanent shocks.

Our contribution is to show the ability of the three-equation NK model to explain the facts. We take a fresh look at the model and show that it supports both a negative and

[1] Given a reasonable parameterization of the three-equation NK model (cf. Section 2).

a positive response of inflation to a 'monetary policy shock', once the definition includes temporary and/or permanent components. We propose a novel accounting procedure to identify shocks similar to Chari, Kehoe, and McGrattan (2007), based on the implied policy functions. We show how the term structure of interest rates helps to identify permanent shocks to interest rates and inflation. The variability at the long-end of the yield curve may reflect changes in uncertainty, concrete policy action and/or by changing expectations (e.g., by unconventional monetary policy). This strategy accounts for the enlarged set of policy instruments of the monetary authority. Based on the identified shocks to inflation and the natural rate, we show that the simple NK model is able to replicate the key macroeconomic variables *and* the term structure of interest rates. In particular, shocks to the interest rate and the natural rate may result into a ZIRP period, and by the same arguments, inflation may rebound while nominal interest rates being kept near-zero values. We show how the continuous-time three-equation NK model emerges as an approximation of the stochastic model with shocks and discuss the differences to the full nonlinear solution. Most importantly, our accounting procedure extends to the nonlinear version of the model with shocks. A nonlinear (and global) solution, which accounts for strong nonlinearities capturing non-normal times, is useful to generate immobile interest rates near-zero and stable quiet inflation with a single shock.

The quest on the ability of the simple NK model to explain the recent episodes has a deeper motivation. We investigate whether financial frictions are required to reconcile the recent facts with the theory. Although formulating and computing medium-scale NK models is important (cf. Christiano, Motto, and Rostagno, 2014; Del Negro, Giannoni, and Schorfheide, 2015), there is need for a parsimonious specification to provide a simple device to develop intuition, conceptualize, and facilitate the way we think about problems in economics. Identifying latent variables gives guidance for a structural interpretation in more elaborated models. This paper merely shows that temporary and permanent shocks in the interest rate (and inflation) enable the simple NK model to replicate the data. It is not intended to identify the primitive sources of shocks. It rather shows that many models are observational equivalent to a simple NK model with both type of shocks. We also provide an analytical investigation of the effects of uncertainty in the nonlinear NK model and show how it affects the natural rate. Our results indeed show that uncertainty shocks are isomorphic to discount factor shocks (Barsky, Justiniano, and Melosi, 2014), so they provide an attractive structural interpretation of the permanent shocks.

Our arguments are motivated by the strong empirical evidence of shifting end-points in the yield curve (cf. Kozicki and Tinsley, 2001; Gürkaynak, Sack, and Swanson, 2005; Bauer and Rudebusch, 2020).[2] There is also empirical evidence in the macro literature

[2] Linking the policy target rates to the long-end of the yield curve is not new and received increasing attention (see Gürkaynak and Wright, 2012, and the references therein). Time-variation in the inflation target is needed to capture the evolution of inflation expectations (cf. Del Negro and Eusepi, 2011).

on time-varying inflation targets (cf. Ireland, 2007; Fève, Matheron, and Sahuc, 2010). Empirical results for the US and Japan is also confirmative of the counteracting effects resulting from transitory and permanent shocks to the interest rate (cf. Uribe, 2017).

In contrast to the ZLB literature, we focus on equilibria with active monetary policy, in which the enlarged set of policy instruments includes long-end target rates (the liquidity trap scenario is studied in Werning, 2012; Cochrane, 2017b; Wieland, 2019). This paper thus fills the gap in the literature by providing an investigation of the simple NK model in non-normal times when the traditional arguments seem to fail. This is highly relevant since the mode of criticism relates to the case that the ZIRP period reflected a binding constraint. We find that changes in longer rates (possibly through unconventional policies) help to explain the recent episodes within the simple NK framework, while nonlinearities play an important role to generate the ZIRP period as a policy choice.

The rest of the paper is organized as follows. First, in Section 2 we study the simple NK model and a novel identification scheme and explore the ability to explain the recent facts. In Section 3 we provide an thorough analysis and present the underlying full nonlinear model by introducing more shocks, and show how near-zero interest rates can be reconciled within the framework, and may result as a policy choice. Section 4 concludes. Further results and illustrations are available in an accompanying web appendix.

## 2. Simplified Framework

In this section we present the continuous-time specification of the standard NK model. This simple framework is used to answer our questions regarding the ability of the model to explain the facts. In the next section we show how the equilibrium dynamics follow from the standard micro-founded rational-expectation solution and shed light on the effects of uncertainty, and potential sources of permanent shocks to interest rates and inflation.

### 2.1. The three-equation NK model

The simplest version of the NK model reads:

$$\mathrm{d}x_t = (i_t - r_t - \pi_t)\mathrm{d}t \tag{1}$$

$$\mathrm{d}\pi_t = (\rho(\pi_t - \pi_t^*) - \kappa x_t)\mathrm{d}t \tag{2}$$

We denote $x_t$ as the output gap (percentage deviations), $i_t$ is the nominal interest rate, $r_t$ is the natural rate, which coincides with the rate of time preference $\rho$, once transitory shocks have abated, $r_t^* = \rho$, and $\pi_t$ is inflation, where $\kappa$ controls the degree of price stickiness with $\kappa \to \infty$ as the frictionless (flexible price) and $\kappa \to 0$ perfectly inelastic (fixed price) limits. This system summarizes the linearized equilibrium dynamics around zero-inflation target $\pi_t^* = 0$ (or full indexation).[3]

The equation (1) follows directly from the consumption Euler equation representing the optimal investment/saving (IS) decision, often referred to as the IS curve, whereas (2) is the NK forward-looking Phillips curve. Solving forward it expresses inflation in terms of future output gaps,

$$\pi_t - \pi_t^* = \kappa \int_t^\infty e^{-\rho(v-t)} x_v \, \mathrm{d}v.$$

Hence, the *current* rate of inflation is forward looking and accounts for future changes in the output gap. In this model it is useful to think of the path of expected future inflation and other variables (e.g., marginal cost) determining events at time $t$.

We close the model by specifying a rule which determines the (equilibrium) interest rates. In this perfect-foresight model both inflation dynamics and the output gap are fully determined by the Taylor rule. In what follows we analyze two alternative setups, which we refer to as the traditional feedback model:

$$i_t = \phi(\pi_t - \pi_t^*) + i_t^*, \quad \phi > 0, \tag{3a}$$

and the partial adjustment model (similar to Sims, 2004; Cochrane, 2017b):

$$\mathrm{d}i_t = \theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\mathrm{d}t, \quad \theta > 0, \tag{3b}$$

which reflects both a response to inflation and a desire to smooth interest rates. The rules (3a) and (3b) show the attitude of the monetary authority towards either the long-run nominal interest rate or the target rate of inflation (one target is isomorphic to the other). In this paper, we consider the inflation target as a policy parameter, but abstract from specifying a specific process. We interpret unexpected changes in target rates as to capture changes in the conduct of monetary policy. Empirically, variations in the long rates are crucial for understanding the dynamics of yields (cf. Bauer and Rudebusch, 2020).

The rule (3b) specifies an explicit time lag between the inflation rate $\pi_t$ and the policy rate $i_t$. The delay will be small if the parameter $\theta$ is large:

$$i_t - i_t^* = \phi\theta \int_{-\infty}^t e^{-\theta(t-k)} (\pi_k - \pi_t^*) \, \mathrm{d}k,$$

which makes $i_t$ a *state* variable, given by past inflation rates. While the rule (3a) may seem simpler, it has some undesirable properties in continuous time. Among others, the clear distinction between inflation (expected future inflation) that the interest rate controls and inflation to which the Fed responds vanishes in continuous time.

Before we can meaningfully study shocks to the interest rate it is important to answer

[3]Note that the appearance of the inflation target $\pi_t^*$ in (2) ensures that the long-run equilibrium coincides with the nonlinear solution (cf. Section A.1).

the question about local determinacy and thus the possibility of sunspot equilibria. We define an active monetary policy if $\phi > 1$ and refer to monetary policy as passive if $\phi < 1$. In what follows we focus on an active monetary policy ensures the existence of a unique locally bounded solution (cf. Appendix A.1.1 and the web appendix).[4]

### 2.2. Term structure of interest rates

Financial market data can shed light on expectations, which is particularly relevant when the standard instrument of monetary policy is not available. In particular, the yield curves contain valuable information about future paths of interest rates and inflation.

Let us consider a nominal (zero-coupon) bond with unity payoff at maturity $N$:

$$P_t^{(N)} = \mathbb{E}_t \left( e^{-\rho N} \lambda_{t+N} / \lambda_t e^{-\int_t^{t+N} \pi_s ds} \right),$$

where $\lambda_t$ is the marginal value of wealth, or the present value shadow price, consistent with equilibrium dynamics of macro aggregates.[5] The equilibrium bond price can be obtained from the the partial differential equation (henceforth *PDE approach*)

$$\theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))(\partial P_t^{(N)} / \partial i_t) - (\partial P_t^{(N)} / \partial N) - i_t P_t^{(N)} = 0.$$

The solution to the pricing equation implies the complete term structure of interest rate for any given interest rate and maturity (we study the term premium in Section 3):

$$y_t^{(N)} \equiv y^{(N)}(i_t) = -\log P_t^{(N)}/N = -\log P^{(N)}(i_t)/N. \tag{4}$$

There are efficient algorithms to solve the fundamental pricing equation.[6] Our strategy is to use collocation, so we approximate the function $P_t^{(N)} \approx \Phi(N, i_t)v$, in which $v$ is an $n$-vector of coefficients and $\Phi$ denotes the known $n \times n$ basis matrix, and can compute the unknown coefficients from a *linear* interpolation equation:

$$\theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\Phi_2'(N, i_t)v - \Phi_1'(N, i_t)v = i_t \Phi(N, i_t)v,$$

or

$$\left(\theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\Phi_2' - \Phi_1' - i_t \Phi\right) v = 0_n,$$

[4] Note that the indeterminacy regions typically depend on the modelling frequency (Hintermaier, 2005). Hence, the findings for the discrete-time model with a presumed timing convention cannot simply be translated to different decision horizons, in particular to the continuous-time limit.

[5] In this simplified framework, we abstract from a term premium to focus on the expectation channel of the NK model. Below we discuss the term premium and the full model with shocks (cf. Section 3).

[6] The traditional *expectation approach* is to simulate the $N$-period ahead distribution (cf. Cochrane, 2005; Gürkaynak, Sack, and Swanson, 2005). While the approach is easy to implement, approximating moments for more state variables and longer maturities easily becomes computationally infeasible.

where $n = n_1 \cdot n_2$ with the boundary condition $\Phi(0, i_t)v = 1_n$. So we concatenate the two matrices and solve the linear equation for the unknown coefficients:

$$\left[ \begin{array}{cc} \Phi_1' - \theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\Phi_2' + i_t\Phi, & \Phi(0, A_t) \end{array} \right] v \;=\; \left[ \begin{array}{c} 0_n \\ 1_n \end{array} \right].$$

For the *feedback model*, with no relevant state variables, we obtain the trivial solution that without shocks the yield curve is flat, $i_t \equiv i_t^* = \rho + \pi_t^*$. This analytical solution of the feedback model, however, is useful for studying the long-end of both the nominal and real yield curves in the partial adjustment model and thus for interpreting the data:

$$\lim_{N\to\infty} y_t^{(N)} = i_t^* = \rho + \pi_t^*, \quad \lim_{N\to\infty} r_t^{(N)} = i_t^* - \pi_t^* = \rho \equiv r_t^*, \tag{5}$$

which show the expectation component of the long-end yields. The inflation target does not affect the long-run risk-free rate, but the nominal rate, $i_t^* = \rho + \pi_t^*$. For comparison with the data we consistently define the model-implied 10-years to maturity yields of (nominal and inflation-protected) zero-coupon bonds $y_t^{(10)}$ and $r_t^{(10)}$, respectively.[7]

## 2.3. Data

Our data is from the Federal Reserve Bank of St. Louis Economic Dataset (FRED), which mainly includes the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index (Core CPI), seasonally adjusted, the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Output gap (HP Filter) obtained from the US Gross Domestic Product from January 1990 through August 2020 (cf. Figure 1).[8] Because our approach does not make any timing assumption, we can use the same policy functions for both monthly and quarterly data (end-of-period figures). We use quarterly data whenever we include the Output gap.[9]

## 2.4. Secular trends and temporary shocks

Having at hand the term-structure implications, we may account for empirical variations in trend inflation and the equilibrium real interest rate (Bauer and Rudebusch, 2020). We extend the three-equation NK model by allowing for permanent shocks to the interest rate (and inflation), i.e., implying variations in the long-end of the yield curves (5).

[7]Figure 2 shows the model-implied 10-years to maturity yield of a zero-coupon bond $y_t^{(10)} = y^{(10)}(i_t)$ and the 10-years to maturity inflation-protected yield $r_t^{(10)} = r^{(10)}(i_t)$.

[8]Federal Reserve Economic Data: https://fred.stlouisfed.org

[9]This paper is *not* about the choice of data series, in particular of choosing the appropriate data set and data frequency. Because our insights do not depend on that choice, we simply use HP filtered data to proxy the output gap (see Kamber, Morley, and Wong, 2018, and Figure E.1 in the web appendix for alternative measures). Recent research suggests the use mixed-frequency and financial market data to identify macroeconomic shocks (among others Christensen, Posch, and van der Wel, 2016).

In this paper, we consider the trend inflation $\pi_t^*$ and the (Wicksellian) natural rate $r_t^*$ as exogenous parametric values.[10] Economically, the inflation target $\pi_t^*$ may be considered a policy instrument, such that a 'target shock' simply reflects (unexpected) changes to $\pi_t^*$. The second source of long-run variation are shocks to the natural rate $r_t^*$. Below we offer some structural interpretation of such shocks beyond the central banks control.

We allow for temporary shocks to the policy rate $i_t$ (monetary policy shock) and the natural rate $r_t$ (preference shock), i.e., implying variations in the short-end of the yield curve. In the simplified framework we compute the perfect-foresight solutions with probability-zero jumps in the interest rate $i_t$. To avoid confusion with the probabilistic model (see Section 3), and to keep notation simple we do not introduce separate variables for probability-zero shocks. A shock simply initializes the dynamic system at a different state (probability-zero jumps in state variables), while the difference to the left-hand limit $t_-$ defines the size of the shock at $t$. Further we introduce an autoregressive shock process $d_t$ with $\rho_d > 0$, which determines the persistence of preference shocks:

$$\mathrm{d}d_t = -\rho_d(d_t - 1)\mathrm{d}t, \tag{6}$$

such that $r_t = r_t^* + \rho_d(d_t - 1)$ defines the 'natural rate' of interest (e.g., Werning, 2012). In the partial adjustment model (3b) which is used below, the interest rate $i_t$ is a state, generating its own dynamics. For the feedback model (3a), we would need to add another exogenous process to introduce (temporary) monetary policy shocks similar to (6).

Our approach is to study whether the simple NK model (traditional parameterization) is able to explain the recent dynamics of macroeconomic aggregates *and* is consistent with the term structure, in particular the expected paths of interest rates and inflation.

### 2.5. The identification of temporary and permanent shocks

So far we have described the three-equation NK model and the sources of temporary and permanent shocks. Our identification of shocks is quite different from the traditional identification schemes.[11] At a conceptual level, we closely follow the idea of business cycle accounting (Chari, Kehoe, and McGrattan, 2007), but also simplify that approach. Using data on key macroeconomic aggregates and the term structure of interest rate we identify temporary and permanent (zero-probability) shocks required to replicate the data. We do not aim to identify the primitive sources of temporary and permanent shocks, rather shed light on potential sources and alternative explanations below (cf. Section 3).

Our identification scheme is intended to investigate to which extent the simple NK

[10] Note that the assumption of constant target rates will not be relevant for our arguments. Alternative approaches such as the regime-switching framework (see Sims and Zha, 2006), or time-varying inflation targets (e.g., Ireland, 2007) would be more realistic, at the cost of more technical details.

[11] Our definition of exogenous shocks coincides with primitive forces that are uncorrelated rather than innovations from a reduced form vector autoregression model (cf. Ramey, 2016; Uribe, 2017).

model is *able* to explain the data and whether the shocks to the macroeconomic aggregates are primarily temporary and/or permanent. This is a useful first step in guiding the construction of detailed models with various frictions.[12] Our accounting procedure can also be applied to the stochastic version of the model (cf. Section 3). This is important, because the equilibrium of the model depends on expectations and, therefore, on the underlying stochastic processes that may generate substantial effects of uncertainty.

Formally, given the model-implied functions (henceforth predictions) and observable variables, our goal is to find a parsimonious set of unobserved state variables, loosely interpreted as 'shocks', making the difference between predictions and actual data zero. Our numerical routines attempt to solve the system of equations by minimizing the sum of squares of the function components. If the equations can be solved up to the numerical accuracy, we say that the NK model is able to explain the given data. Alternatively, a minimizer can be used to provide the best possible fit to the given data, leaving aside the decision whether the residuals are negligible. An empirical model therefore consists of the triple of predictions, observed states and latent state variables (shocks).

### 2.5.1. A model with temporary shocks only

For illustration, given the dynamics in (6), the theoretical NK model implies inflation $\pi_t = \pi(i_t, d_t)$, output gap $x_t = x(i_t, d_t)$, together with the 10-year yields to maturity of nominal bonds, $y_t^{(10)} = y^{(10)}(i_t, d_t)$, and inflation-protected bonds $r_t^{(10)} = r^{(10)}(i_t, d_t)$, as functions of the interest rate and the (temporary) natural rate shock.[13] Consider the empirical model for inflation $\pi_t$ (together with the output gap $x_t$), the observed interest rate $i_t$, and the latent natural rate shock $d_t$. In this specification we allow for temporary shocks only and use monthly (quarterly) data. We solve for the latent shock process and thereby uncover the state $d_t$ (or the natural rate $r_t$), and infer the interest rate shocks $i_{t+\Delta} - \mathbb{E}_t(i_{t+\Delta})$ (or monetary policy shocks), for the set of predictions, parameterization, and the observed data.[14] Hence, we extract the natural rate required to replicate the interest rate (Fed Funds), the inflation rate (Core CPI), and the output gap (Output gap) for the fixed parameters $\rho = 0.03$, $\kappa = 0.8842$, $\phi = 4$, $\theta = 0.5$, $\pi_t^* = 0.02$, and $\rho_d = 0.4214$. This particular set of parameters is within plausible estimates discussed in the literature. For the ability question, we need at least one admissible set from the parameters able to replicate the data. If this set was empty, any estimation strategy would be ill-conditioned. This parameterization implies the (Wicksellian) natural rate $r_t^* = 0.03$.

[12] Several authors draw the connections that financial market frictions can trigger a drop in the natural rate of interest (Eggertsson and Krugman, 2012; Del Negro, Eggertsson, Ferrero, and Kiyotaki, 2017).

[13] We solve the boundary value problem for the function $x : [0, T \mapsto \mathbb{R}^k]$ that satisfies the equations (1), (2), (3b), and (6) given the initial condition for the $k$ state variables together with the TVCs assuming that variables approach their steady state values for $T \to \infty$ (cf. Section 3 for details).

[14] Our definition of a monetary policy shock, $i_{t+\Delta} - \mathbb{E}_t(i_{t+\Delta})$, follows the interpretation of 'shocks' as the unexpected endogenous response of monetary policy to a primitive shock (cf. Ramey, 2016). Only the limiting case of $\Delta \to 0$ generates exogenous shocks to $i_t$ that are uncorrelated with other shocks.

Two remarks are noteworthy. First, usually the number of shocks should not exceed the number of observed variables. In the example above we have two shocks: one shock to the natural rate and one shock to replicate the interest rate (monetary policy shock); and two observed variables: the inflation rate and the interest rate. Second, the primitive source of (temporary) shocks to the natural rate is irrelevant for the natural rate. Though affecting the predictions, our conclusions are consistent with other types of shocks.

Based on the identified natural rate $r_t$ (see Figures 3 and 4), we find that the model is able to explain interest rate and inflation dynamics, but dramatically fails to generate sufficient variability in 10-year treasury rates (see Figures 5 and 6). In other words, the expectations are not consistent with the data and/or the model misses relevant variations in term premia, or any other 'permanent' shock (see Section 3).[15] If the empirical model includes the output dynamics, the NK model with temporary shocks to the natural rate is *not* able to explain about 20 percent of the data for the given parameterization, so we use a minimum distance approach (compare to Figure E.5 in the web appendix).

One may argue that adding a second temporary shock (fiscal policy, technology, etc.) would restore the ability the simple NK model to explain the macroeconomic dynamics. But ultimately the model seems not adequate, as revealed by the financial market data on interest rates from the yield curve (see Figure 6). Both the implied inflation rates and implied 10-year treasury rates indicate that the data calls for more variations at longer horizons. For example, the observed large and persistent deviations shed light on the fact that the predictions for the interest rates have been substantially lower than the realized interest rates over the first half of the sample, but too high in the second half, so the model misses the long-run trend (cf. Bauer and Rudebusch, 2020).

### 2.5.2. A model with temporary and permanent shocks

Suppose that in addition we allow for permanent shocks to the interest rate and inflation (by allowing time-variability in $r_t^*$ and $\pi_t^*$) in order to account for the variability of the long-end of the nominal and the real yield curves. We obtain predictions for the inflation rate $\pi_t = \pi(i_t, d_t; r_t^*, \pi_t^*)$, output gap $x_t = x(i_t, d_t; r_t^*, \pi_t^*)$, together with the 10-year yields to maturity of nominal bonds, $y_t^{(10)} = y^{(10)}(i_t, d_t; r_t^*, \pi_t^*)$, and inflation-protected bonds $r_t^{(10)} = r^{(10)}(i_t, d_t; r_t^*, \pi_t^*)$, as functions the interest rate and the (temporary) natural rate shock, respectively the parameters of the Wicksellian natural rate and the inflation target. Consider then the empirical model for inflation $\pi_t$ (together with the output gap $x_t$), the 10-year yields to maturity of nominal bonds, $y_t^{(10)}$, and inflation-protected bonds, $r_t^{(10)}$, the observed interest rate $i_t$, two latent shocks to the natural rate, $r_t = r_t^* + \rho_d(d_t - 1)$, and the shocks to the inflation target $\pi_t^*$. In this specification we allow for temporary

[15] Christiano, Motto, and Rostagno (2014) include the slope of the term structure of interest rates (i.e., the difference of the long-term bond and the federal funds rate), by including an exogenous measurement error shock on the long-term bonds, which they interpret as a term premium shock.

and permanent shocks using monthly (quarterly) data. We solve for the latent shocks and thereby uncover the state $d_t$ (or the natural rate $r_t$) together with the time-varying parameters $r_t^*$ and $\pi_t^*$, and infer the monetary policy shocks $i_{t+\Delta} - \mathbb{E}_t(i_{t+\Delta})$ for the set of predictions, parameterization, and observed data. Hence, we extract the natural rate and time-varying parameters required to replicate the interest rate (Fed Funds), the inflation rate (Core CPI), the output gap (Output gap), and the 10-Year Treasury rates (10Y Govt and 10Y TIPS) for the remaining fixed parameters $\phi = 4$, $\theta = 0.5$, and $\rho_d = 0.4214$.[16]

Based on the identified natural rate $r_t$, $\pi_t^*$ and $r_t^*$ and $i_t$ (see Figure 7), we find that the model is able to replicate interest rate, inflation dynamics, the slope of the yield curves, and output (see Figures 9 and 10). Even when the empirical model includes the output dynamics, the NK model with temporary and permanent shocks to the natural rate and inflation *is* able to explain the data for the given parameterization, so actually we solve the system of equations with four shocks and five observed variables. From the identified series of shocks, the permanent shocks to the natural rate seems the key to explain the inflation rebound during the zero-interest-rate policy period (ZIRP), whereas the inflation target is closely centered around 2 percent at least since 2003 for which TIPS is available. There seems to be a rational for a (perceived) permanent shock to the inflation target at begin of the ZIRP episode, which in part explains the sharp cut in interest rates at the end of 2007. The implied monetary policy shock allows us to shed light to which extent the interest rate cut has been interpreted as permanent or transitory.

One drawback is that TIPS are available only from 2003Q1. In an alternative empirical specification we allow for variations in $r_t^*$ keeping the inflation target constant ($\pi_t^* = 0.02$), and identify permanent shocks without using TIPS. In this model, we identify the natural rate $r_t$, together with the shocks for $r_t^*$ to replicate the observed data, by including the nominal 10-Year Treasury rates (10Y Govt). Still, the three-equation NK model is able to replicate interest rates, inflation dynamics, and both long-end yields, even though the inflation-protected 10-Year Treasury rates (10Y TIPS) are not used for identification. For most of observations output dynamics can be replicated (cf. Figures E.10 and E.11). The bottom line is that both temporary and permanent shocks are required (also sufficient) for the ability of the simple NK model with $\phi > 1$ to explain the facts.

### 2.6. Discussion

In the previous section we studied the ability of the simple NK model to explain the recent episodes, including the financial crisis episodes. This section sheds light on some anecdotal evidence, which has been used to argue that the traditional arguments are flawed.

We are now in the position to shed light on the cut in interest rates from 5.25 percent by the end of 2007Q2 to 0.16 percent by the end of 2008Q4 (about −500 basis points) and

[16] As shown below, the parameter $\kappa$ is a function of $\rho$ and thus the slope of the Phillips curve varies by allowing time-variation in the long-end of the yield curve (see Section 3 for details).

the three-equation NK model with active monetary policy. We showed that the model is able to explain the dynamics of key macroeconomic variables *and* the term structure of interest rates. Based on the identified shocks, we can now study to which extent the cut was a response of monetary policy to other shocks. Let us use the figures for the model with temporary shocks only and with both temporary and permanent shocks.

In the three-equation NK model with temporary shocks only, a back-of-the-envelope calculation says that the natural rate shock was −420 basis points (bp) and the monetary policy shock, which includes the policy response to this shock, was about −580 bp.[17] This implies that more that 3/4 of the sharp decline in interest rates from 2007Q2 through 2008Q4 is attributed to a temporary *response* of monetary policy.

Allowing for temporary and permanent shocks, the same calculation imply that the natural rate shock was −230 bp and the monetary policy shock, which includes the policy response to the natural rate shock, was about −540 bp.[18] This interest rate cut partly has been interpreted as a permanent shock by about −130 bp, such that about 1/2 of the interest rates cut from 2007Q2 through 2008Q4 can be attributed to changes in monetary policy (of which −100 bp permanent) in *response* to the negative natural rate shock that brought the Wicksellian natural rate to decline by about −30 bp.

We now focus on three episodes brought forward by economists, and discussed in the literature. First, to the unaided eye, the data suggests a sign reversal of the traditional tradeoff in the period 2001-2007, supporting the hypothesis that inflation and interest rates are positively related. If anything, inflation decreased in response to the interest rate cuts. Second, in the subsequent period from 2007 the Fed Funds rate has remained near zero until end of 2015, to which we refer as the zero-interest-rate policy (ZIRP) period, but inflation kept stable and quiet (cf. Cochrane, 2017a). Despite interest rates near zero through 2015, inflation rebounded in 2011, with about the same pattern as before. While the short rate seems immobile over that episode, the long-end of the yield curve has considerable variation and declines over time. From the expectation hypothesis, we may also read this as changes of market perceptions about future interest rates and/or monetary policy. Third, there is anecdotal evidence of apparent term structure anomalies between 2004Q2 and 2005Q2: The Fed Funds rate increased by 200 bp, but the 10Y Govt decreased by about 70 bp (cf. Backus and Wright, 2007). So can we explain the recent episodes and term structure anomalies within the three-equation NK model?

Let us now turn to these particularities of the data. We may use the NK model to

[17] More precisely, the natural rate declined from 5.00% to −0.17% (or −483 bp). The expected natural rate by the end of 2008Q4 was 4.06%, so the shock to the natural rate was −423 bp. Similarly, the expected interest rate by the end of 2008Q4 was 5.93%, so the monetary policy shock was −577 bp.

[18] Here, the natural rate dropped from 5.88% to 1.99% (or −389 bp). The expected natural rate by the end of 2008Q4 was 4.32%, so the shock to the natural rate was −233 bp. Similarly, the expected interest rate by the end of 2008Q4 was 5.55%, so the monetary policy shock was −539 bp. Moreover, the inflation target declined from 1.38% to 0.40% (or −98 bp), and the Wicksellian rate declined from 2.54% to 2.23% (or −31 bp), so the long-term interest rate dropped by −129 bp.

interpret the episodes: (i) with an apparent sign reversal (2001-2007), (ii) including a zero-interest-rate policy with an inflation rebound despite near-zero interest rates (2007-2015), and (iii) including an apparent term structure anomaly (2004-2005).

#### 2.6.1. Sign reversal

While the academic discourse about the effects of the nominal interest rate on the inflation rate has some tradition in macroeconomics, motivated by the 'price puzzle', it received public attention in 2008, when the interest rates in the US (followed by the ECB in 2014) hit essentially zero. Consider the period 2001-2007, right before the financial crisis.

In Jan 2001 the Fed Funds rate was at 6.5 percent (6.40%), the 10Y Govt at 5 percent (5.12%). In Oct 2007 the Fed Funds rate was slightly below 5 percent (4.94%), the 10Y Govt at 4.5 percent (4.59%). In the meantime, the Fed Funds rate has been sharply decreased and raised to and from 1 percent. Over the same period, the Core CPI inflation followed a similar $\vee$ pattern and decreased slightly from 2.5 percent (2.57%) to values around 2 percent (2.10%). When the Fed Funds rate dipped at 1 percent (0.98%) in Dec 2003, inflation also had its lowest value of 1 percentage point (1.09%) with 10Y Govt at 4 percent (4.27%). Can we reconcile this pattern with the NK model?

If we interpreted the $\vee$ pattern as two consecutive temporary monetary policy shocks, the NK model predicts that inflation should have followed a counterfactual $\wedge$ pattern.[19] A transitory (negative) monetary policy shock of 500 bp would imply inflation to increase by about 250 bp in 2003. However, the same transitory shock together with a (negative) permanent shock of 150 bp would account for the observed pattern for inflation and would also predict the decline in yields to longer maturities (see Figures E.26 and E.27). Similarly, a (positive) temporary monetary policy shock of 400 bp together with restoring the announced inflation target rate in 2007 has the opposite effect and may have generated the observed $\vee$ pattern of Fed Funds, 10Y yields and Core CPI inflation in the period.

Let us now illustrate the dynamics through the identified shocks of the three-equation model (cf. Figures 7 and 8). Suppose that by the end of 2000Q4 the inflation target was at 2 percent (we assume $\pi_t^* = 0.02$ as TIPS are not available), the natural rate was at 6 percent (5.85%). Until Jan 2004 the natural rate dropped by about 825 bp, mainly driven by a large negative preference shock by the end of 2003Q4 ($-13.51\%$), with the Wicksellian rate at 3.5 percent (3.28%), and the inflation target close to 2 percent (2.18%). In Oct 2007, the natural rate was back to 6 percent (6.10%), the Wicksellian rate around 2 percent (2.12%) with the inflation target at 1 percent (1.16%). Based on the identified shocks, we do not only replicate the observed $\vee$ pattern of the Fed Funds, the 10Y Govt, and the Core CPI inflation, we also obtain predictions for output and the yield curves in line with observed data (cf. Figures 11 and 12 and the web appendix).

[19] A 'monetary policy shock' is defined as the discrete change in $i_t$, which is $i_t \equiv \lim_{\Delta \to 0}(i_{t+\Delta} - \mathbb{E}_t(i_{t+\Delta}))$ in the remaining analysis (transitional dynamics are shown in an accompanying web appendix).

To summarize, the observed pattern indeed can be reconciled with the simple NK model, when allowing for a ∨ pattern to either the inflation target or the natural rate. From the identified shocks, the latter explanation seems more plausible.

### 2.6.2. ZIRP period and inflation rebound

We next consider the zero-interest-rate policy (ZIRP) period 2007-2015, right after the start of the financial crisis from 2007Q3 until the 'liftoff' in Dec 2015, with the end-point marking the start of the Fed's 'normalization' of monetary policy (Williamson, 2016).

In Oct 2007, the Fed Funds rate was at 5 percent (4.94%), the 10Y Govt at 4.5 percent (4.59%), while in Jan 2011 the Fed Funds rate was at 0.25 percent (0.18%) and the 10Y Govt at 3.25 percent (3.30%) . Over the same period, the Core CPI inflation decreased from 2 percent (2.10%) to values way below the announced target rate around 0.5 percent (0.66%) in Jan 2011, and then bounced back in Oct 2011 to values around the announced target of 2 percent (1.99%). At a first glance, the cut at the beginning of the financial crises looks pretty much like the sharp decrease during the 2001-2003 period. This time, however, the (short-term) nominal interest rate was quite close to zero and did not return to higher values for a while. Can we generate a ZIRP period within the NK model?

For a hypothetical interest rate cut by $-475$ bp, with the inflation target of 2 percent, we should have expected inflation rates of more than 4 percent. But Core CPI inflation *declined* from values above 2 percent to around 0.5 percent by the end of 2010Q4, and then rebounded to 2 percent by the end of 2011Q3. In the 'inflation target rate shock' story, the sharp decrease could reflect a drastic drop in the perceived target rate by about $-200$ bp and then returns to values around the announced target. Indeed, inflation would jump to values close to zero and then rebound, but it does *not* explain the persistent near-zero values of the interest rate. Although we can replicate the observed variables at least on impact (including the nominal yield curve), and inflation rates eventually approaching zero, the three-equation NK model predicts a counterfactual strong tendency of the interest rate to revert back to the steady state. Hence, we need another drag on the natural rate that implies a prolonged period of near-zero interest rates. Our simulation results confirm this conjecture: Adding a negative preference shock of roughly $-10$ percent next to the monetary policy shocks helps to fix the yield curve and inflation, but does not generate a ZIRP period either. The inflation response is too strong with negative implications for the fit of 10-year yields. Even with higher persistence the assumed OU process (6) would *not* imply that interest rates do remain close to zero. To explain the ZIRP episode with a *single* shock we would need to modify shock dynamics.

From the identified shocks we find that the inflation target was already relatively low in 2007Q3 such that an increase of the inflation target by 50bp from 1 percent (1.16%) to 1.5 percent (1.78%) in 2010Q4 was indeed accompanied by a substantial decline in the natural rate of about $-870$ bp (cf. Figures 7 and 8). This decline was due to a large negative preference shocks from 2007Q3 (9.44%) to 2009Q4 ($-18.86\%$), and 2010Q4 ($-11.70\%$), which dragged the natural rate to values around $-5$ percent ($-5.14\%$), and $-2.5$ ($-2.60\%$), respectively, and stayed there until the end of 2015Q4 (see Figure E.7). Behind the scenes, however, the inflation target increased further back to 2 percent (2.19%) accompanied by a decline of the Wicksellian natural rate by $-150$ bp to about 0.5 percent (0.66%) in 2011Q3. Because at the same time temporary preference shocks accommodated ($-7.54\%$), the natural rate remained at $-2.5$ percent ($-2.52\%$). To conclude, the shocks replicate the key macroeconomic aggregates and the yield curves over the ZIRP period, but we would need a series of negative shocks to generate this pattern.

### 2.6.3. Term structure anomalies

The discussion on temporary shocks vs. permanent shocks has shown that it is important to consider both, the short and the longer-end of the yield curve in order to interpret the data. Some anecdotal evidence suggests that such shocks may arise simultaneously. If a monetary policy shock is accompanied by a preference shock, some of the 'anomalies' observed in the data arise already in the three-equation NK model.

Let us consider the period between 2004 and 2005, when a rotation in the yield curves gave rise to what Alan Greenspan's called a 'conundrum' (cf. Backus and Wright, 2007). By the end 2004Q2, the Fed Funds rate was at 1 percent (1.03%), the 10Y Govt at 4.5 percent (4.62%), while by the end of 2005Q2, the Fed Funds rate was at 3 percent (3.04%), the 10Y Govt at 4 percent (3.94%). Over the same period, Core CPI inflation increased slightly from below its target rate of 2 percent (1.87%) to about 2 percent (2.03%). The 'conundrum' is that the policy rate increased by 200 bp, but the 10-year yield *decreased* by about 50 bp. Can we reconcile the rotation of the yields, to which we refer as term structure 'anomaly', with the three-equation NK model?

Given the previous discussion, we may conjecture that a positive interest rate shock was accompanied by a negative shock to the natural rate, keeping the inflation target about the same level, such that the negative relationship between interest rates and inflation prevails (with a tendency to revert back to the target rate). If we simulated an interest rate shock of 200 bp which is accompanied by a negative preference shock of about $-10$ percent, both shocks would generate the rotation in the yields as observed. While a rotation in the nominal yield curve could also be obtained by a negative inflation target shock, two reasons speak against this hypothesis for the period 2004-2005: First, if anything, we would expect that a *rise* in nominal interest rates may trigger a *rise* in the inflation target rate. Second, the predicted real yield curve would not show a rotation as observed in the data. Hence, a rotation in both yield curves suggests that the monetary policy shock was accompanied by a shock to the natural rate.

From the identified shocks, the interest rate shock of 200bp was indeed accompanied by an increase of the natural rate by about 380 bp, which mainly is attributed to a reversion of the temporary component from values around $-15$ percent ($-13.56\%$) to values around $-1$ percent ($-1.37\%$). This replicates the observed rotation in the yields together with the dynamics of key macroeconomic aggregates (cf. Figures 13 and 14).

### 2.7. Open questions

So the bottom line is a partial remedy of the NK model to interpret the data. We show how permanent shocks (loosely interpreted as unconventional policy), and shocks to the natural rate in addition to the traditional policy instrument improves the ability of the model to explain the facts. Hence, the three-equation NK model helps us to organize our thoughts and abandon the model might be too shortsighted: Allowing the shocks to have transitory and permanent components, we may explain the $\vee$ pattern of the Fed Funds rate *and* Core CPI inflation (and the Output gap) in the data. These predictions are also in line with the predictions for the yield curve. It also helps to explain the ZIRP period and the inflation rebound in 2011, while interest rates being immobile and near-zero.

Because the three-equation NK model without a binding ZLB fails to replicate the observed pattern during the ZIRP period with a single shock, it seems that the shock dynamics are *not* consistent with the assumed shock process. Based on the identified shocks for the alternative scenarios, respectively, we would need a large shock to the natural rate *and* that this shock keeps the natural rate negative for a while, before eventually reverting back to its steady state. But there is an elephant in the room: The three-equation NK model is unable to account for nonlinearities.

Perhaps we need to distinguish between normal times and non-normal times, where the dynamics are different from those at the intended equilibrium point? This is what we learn from Brunnermeier and Sannikov (2014): In normal times, the equilibrium system is near the steady state, where the system is resilient to most shocks near the steady state. Unusually large shocks, however, may induce completely different dynamics of macro aggregates. Once in a crisis state (non-normal times), also smaller shocks are subject to amplification. A nonlinear framework may be an alternative interpretation in which a single preference shock accounts for the ZIRP period. In what follows, we set up a parsimonious model, where the dynamics of large negative shocks are different from those around the steady state, at which the model is observational equivalent to (6).

In the remaining paper, we formulate and solve the nonlinear version of the NK model. We show that an alternative shock process, which is observational equivalent in normal times (with small shocks), has quiet different dynamics in non-normal times (large shocks). We also allow for stochastic shocks and show how uncertainty shocks will affect the natural rate and the long-end of the yield curves even if the inflation target rate is constant.

## 3. Nonlinear New Keynesian Model with Shocks

We describe now the environment for our investigation. It is the continuous-time version of the standard NK model (cf. Woodford, 2003). We summarize the equilibrium dynamics, show how to compute impulse responses, compute the effects of uncertainty, and how to solve the model in the policy function space. Throughout the paper, we keep the nonlinear structure of the model, which turns out to be quite relevant for non-normal times when considering large shocks and/or large deviations from the point of approximation.

### 3.1. The model

The basic structure of the model is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, who uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good, and face the constraint that they can only adjust the price following Calvo's pricing rule (Calvo, 1983). Finally, there is a monetary authority that fixes the short-term nominal interest rate through open market operations with public debt, and a fiscal authority that taxes and consumes. We introduce four stochastic shocks, one to preferences (which can be loosely interpreted as a shock to aggregate demand, temporarily affecting the real interest rate), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy. For simplicity, we do not explicitly model a shock to the inflation target, which is considered a policy instrument under the discretion of the central bank.

#### 3.1.1. Households

There is a representative household in the economy that maximizes the following lifetime utility function, which is separable in consumption, $c_t$ and hours worked, $l_t$:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \mathrm{d}t, \quad \psi > 0, \tag{7}$$

where $\rho$ is the subjective rate of time preference, $\vartheta$ is the inverse of Frisch labor supply elasticity, and $d_t$ is a preference shock, with $\log d_t$ following an Ornstein-Uhlenbeck (OU) process (the continuous-time analog of a first-order autoregression):

$$\mathrm{d}\log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}. \tag{8}$$

The process $B_{d,t}$ is a standard Brownian motion, such that by Itô's lemma:

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}.$$

Below, for this shock and the other exogenous stochastic processes, we will use both the formulation in level and in logs depending on the context and ease of notation.

Let $a_t$ denote real financial wealth, the household's real wealth evolves according to:

$$\mathrm{d}a_t = \left((i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t\right)\mathrm{d}t, \tag{9}$$

in which $i_t$ is the nominal interest rate on government bonds, $\pi_t$ the rate of inflation of the price level $p_t$ (or price of the consumption good), $w_t$ is the real wage, $T_t$ is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy.

### 3.1.2. The final good producer

There is one final good produced using intermediate goods with the following production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{10}$$

where $\varepsilon$ is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (10), taking as given all intermediate goods prices $p_{it}$ and the final good price $p_t$. Hence, the input demand functions associated with this problem are:

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t \qquad \forall i,$$

and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon}\mathrm{d}i\right)^{\frac{1}{1-\varepsilon}} \tag{11}$$

is the (aggregate) price level.

### 3.1.3. Intermediate good producers

Each intermediate firm produces differentiated goods out of labor using:

$$y_{it} = A_t l_{it},$$

where $l_{it}$ is the amount of the labor input rented by the firm and where $A_t$ follows:

$$\mathrm{d}\log A_t = -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}. \tag{12}$$

Therefore, the marginal cost of the intermediate good producer is the same across firms:

$$mc_t = w_t/A_t. \tag{13}$$

The monopolistic firms engage in infrequent price setting à la Calvo. We assume that intermediate good producers reoptimize their prices $p_{it}$ only when a price-change signal is received. The probability (density) of receiving such a signal $h$ periods from today is assumed to be independent of the last time the firm got the signal, and to be given by:

$$\delta e^{-\delta h}, \quad \delta > 0.$$

Thus $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)} \, \mathrm{d}h = 1 - \left(-e^{-\delta(\tau-t)} + 1\right) = e^{-\delta(\tau-t)}. \tag{14}$$

A fraction of firms will receive the price-change signal per unit of time. All other firms cannot reoptimize their price, but (partially) index their price to $\pi_t^*$:

$$\mathrm{d}\tilde{p}_{it} = \chi \pi_t^* \tilde{p}_{it} \, \mathrm{d}t$$

Indexation is controlled by the parameter $\chi \in [0, 1]$. This implies that if a firm cannot change its price at $t$ for a period length of $\tau - t$, its price at $\tau$ is $p_{i\tau} = p_{it} e^{\int_t^\tau \chi \pi_s^* \, \mathrm{d}s}$. Note that the higher the parameter $\delta$, the lower price rigidities, in the frictionless case $\delta \to \infty$. Hence, for $\delta < \infty$ prices are set to maximize the expected discounted profits:

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-(\rho+\delta)(\tau-t)} \left( \frac{p_{i\tau}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau} \right) \mathrm{d}\tau \quad \text{s.t. } y_{i\tau} = \left( \frac{p_{i\tau}}{p_\tau} \right)^{-\varepsilon} y_\tau,$$

or

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-(\rho+\delta)(\tau-t)} \left( \left( \frac{p_{it}}{p_\tau} \right)^{1-\varepsilon} e^{\int_t^\tau (1-\varepsilon)\chi \pi_s^* \, \mathrm{d}s} y_\tau - mc_\tau \left( \frac{p_{it}}{p_\tau} \right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon \chi \pi_s^* \, \mathrm{d}s} y_\tau \right) \mathrm{d}\tau$$

After dropping constants, we may write the first-order condition as:

$$\begin{aligned} &\mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} (1-\varepsilon) \left( \frac{p_t}{p_\tau} \right)^{1-\varepsilon} p_{it} e^{\int_t^\tau (1-\varepsilon)\chi \pi_s^* \, \mathrm{d}s} y_\tau \mathrm{d}\tau \\ &+ \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} mc_\tau \varepsilon \left( \frac{p_t}{p_\tau} \right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon \chi \pi_s^* \, \mathrm{d}s} p_t y_\tau \mathrm{d}\tau = 0 \end{aligned}$$

We may write the first-order condition as:

$$p_{it} x_{1,t} = \frac{\varepsilon}{\varepsilon - 1} p_t x_{2,t} \quad \Rightarrow \quad \Pi_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,t}}{x_{1,t}}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where we define the auxiliary variables (interpreted as expected discounted marginal revenue and marginal costs):

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} e^{\int_t^\tau (1-\varepsilon)\chi\pi_s^* \, \mathrm{d}s} y_\tau \mathrm{d}\tau, \tag{15}$$

$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon\chi\pi_s^* \, \mathrm{d}s} y_\tau \mathrm{d}\tau. \tag{16}$$

Both variables are forward looking (or jump variables) and determined in equilibrium.

Differentiating $x_{1,t}$ with respect to time gives:

$$\mathrm{d}x_{1,t} = ((\rho + \delta + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t) \, \mathrm{d}t \tag{17}$$

in which the rate of inflation $\pi_t = \mathrm{d}p_t/p_t$. Accordingly:

$$\mathrm{d}x_{2,t} = ((\rho + \delta - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - \lambda_t mc_t y_t) \, \mathrm{d}t \tag{18}$$

Note that we assume that the dynamics of the inflation index does *not* contribute to the dynamics of the auxiliary variables (taken as parametric to the firm).

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^t \delta e^{-\delta(t-\tau)} \left(p_{i\tau} e^{\int_\tau^t \chi\pi_s^* \, \mathrm{d}s}\right)^{1-\varepsilon} \mathrm{d}\tau,$$

making the price level $p_t$ a predetermined variable at time $t$, its level being given by past price quotations (Calvo's insight). Differentiating with respect to time gives:

$$\mathrm{d}p_t^{1-\varepsilon} = \left(\delta p_{it}^{1-\varepsilon} - (\delta - (1-\varepsilon)\chi\pi_t^*) p_t^{1-\varepsilon}\right) \mathrm{d}t$$

and

$$\frac{1}{\mathrm{d}t} \mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\, p_t^{-\varepsilon} \frac{\mathrm{d}p_t}{\mathrm{d}t}.$$

Then

$$(1-\varepsilon)\ \mathrm{d}p_t = \left(\delta p_{it}^{1-\varepsilon} p_t^{\varepsilon-1} - (\delta - (1-\varepsilon)\chi\pi_t^*)\right) p_t$$

which implies

$$\pi_t - \chi\pi_t^* = \frac{\delta}{1-\varepsilon} \left((\Pi_t^*)^{1-\varepsilon} - 1\right) \tag{19}$$

Differentiating (19) with respect to time, we obtain the inflation dynamics as:

$$\mathrm{d}(\pi_t - \chi\pi_t^*) = -(\delta + (1-\varepsilon)(\pi_t - \chi\pi_t^*)) \left(\pi_t - \chi\pi_t^* + (mc_t/x_{2,t} - 1/x_{1,t})\lambda_t y_t\right) \mathrm{d}t, \tag{20}$$

which is interpreted as the NK Phillips-curve.

### 3.1.4. The government problem

We assume that the government sets the nominal interest rate $i_t$ through open market operations according to two alternative setups, i.e., the feedback model:

$$i_t - i_t^* = \phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1), \quad \phi_\pi > 0, \ \phi_y \geq 0, \tag{21a}$$

or the partial adjustment model:

$$\mathrm{d}i_t = \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\mathrm{d}t + \sigma_i \mathrm{d}B_{i,t}, \quad \theta > 0, \tag{21b}$$

which includes a response to inflation and output, and a desire to smooth interest rates. Similar to equation (3b), the rule in (21b) specifies a time lag between the inflation rate and the interest rate, and allows for an output response and monetary policy shocks.

The coupon payments of the government treasury bills $T_t^b = -i_t a_t$ are financed through lump-sum taxes. Suppose transfers finance a given stream of government consumption expressed in terms of its constant share of output, $s_g s_{g,t}$, with a mean $s_g$ and a stochastic component $s_{g,t}$ that follows another Ornstein-Uhlenbeck (OU) process,

$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t}\mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \tag{22}$$

such that

$$g_t - T_t^b = s_g s_{g,t} y_t - T_t^b \equiv -T_t.$$

### 3.1.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t = c_t + g_t. \tag{23}$$

In other words, there is no possibility to transfer the output good intertemporally. With this value, the demand for each intermediate good producer is

$$y_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i. \tag{24}$$

Using the production function we may write:

$$A_t l_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}.$$

Integrate both sides:

$$A_t \int_0^1 l_{it}\mathrm{d}i = (c_t + g_t)\int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i$$

to get an expression:

$$c_t + g_t = y_t = \frac{A_t}{v_t} l_t,$$

in which we define:

$$v_t = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i$$

as the aggregate loss of efficiency induced by price dispersion of the intermediate goods. Similar to the price level, $v_t$ is a predetermined variable (Calvo's insight):

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} e^{-\varepsilon \int_\tau^t \chi \pi_s^* \, \mathrm{d}s} \mathrm{d}\tau. \tag{25}$$

Differentiating this expression with respect to time gives:

$$\mathrm{d}v_t \;=\; \left(\delta \left(\Pi_t^*\right)^{-\varepsilon} + \left(\varepsilon(\pi_t - \chi \pi_t^*) - \delta\right) v_t\right) \, \mathrm{d}t. \tag{26}$$

Finally, as shown in the appendix, in equilibrium aggregate profits can be written as a function of other variates:

$$F_t = (1 - mc_t v_t) y_t. \tag{27}$$

### 3.2. The flexible-price case

An important benchmark solution to the NK model is the case where prices become more flexible. In the frictionless limit, $\delta \to \infty$, the firms set prices to maximize profits:

$$\max_{p_{it}} \left(\frac{p_{it}}{p_t} y_{it} - mc_t y_{it}\right) \quad \text{s.t. } y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t,$$

After dropping constants, we may write the first-order condition as:

$$\frac{p_{it}}{p_t} = \frac{\varepsilon}{\varepsilon - 1} mc_t$$

Hence, in the flexible-price case, all firms set prices $p_{it} = p_t$. Given the analytical value of marginal costs, we compute optimal consumption and hours from (34):

$$c_t = (1 - s_g s_{g,t})^{\frac{\vartheta}{1+\vartheta}} A_t((\varepsilon - 1)/(\varepsilon \psi))^{\frac{1}{1+\vartheta}} \tag{28}$$

such that $l_t = ((\varepsilon - 1)/((1 - s_g s_{g,t})\varepsilon\psi))^{\frac{1}{1+\vartheta}}$.

### 3.3. Equilibria with price stickiness

For the case of $\delta < \infty$, we obtain the first-order conditions by defining the state space $U_z \subseteq \mathbb{R}^n$ and the control region $U_x \subseteq \mathbb{R}^m$, the *reward function* $f : U_z \times U_x \to \mathbb{R}$, the *drift function* $g : U_z \times U_x \to \mathbb{R}^n$, and the *diffusion function* $\sigma : U_z \to \mathbb{R}^{n \times n}$. Given our description of the problem, we define the household's value function:

$$V(\mathbb{Z}_0; \mathbb{Y}_0) \equiv \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} f(\mathbb{Z}_t, \mathbb{X}_t)\, \mathrm{d}t,$$

in which $\mathbb{Z}_t \in U_z$ denotes the $n$-vector of states, $\mathbb{X}_t \in U_x$ denotes the $m$-vector of controls, and $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$ is a vector of variates to be determined in equilibrium as a function of the state variables, but taken as parametric by the representative household,

$$s.t. \quad \mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\, \mathrm{d}t + \sigma(\mathbb{Z}_t)\, \mathrm{d}B_t,$$

where $B_t$ is a given vector of independent standard Brownian motions. The instantaneous covariance matrix of $\mathbb{Z}_t$ is $\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top$, which may be less than full rank $n$.

In particular, the vector of state variables is $\mathbb{Z}_t = (a_t, i_t, v_t, d_t, A_t, s_{g,t})^\top$ and equilibrium variables $\mathbb{Y}_t = (y_t, mc_t, w_t, \pi_t, x_{1,t}, x_{2,t}, \Pi_t^*, \lambda_t, T_t, \digamma_t)^\top$ to be determined endogenously, and $\mathbb{X}_t = (c_t, l_t)^\top$ is the vector of controls. In our case, the *reward function* reads:

$$f(\mathbb{Z}_t, \mathbb{X}_t) = d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}.$$

From the discussion above, we define the *drift function* (with partial adjustment):

$$g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t) = \begin{bmatrix} (i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t \\ \theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_t^*) \\ \delta\left(\Pi_t^*\right)^{-\varepsilon} + \left(\varepsilon(\pi_t - \chi\pi_t^*) - \delta\right) v_t \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2)d_t \\ -(\rho_A \log A_t - \frac{1}{2}\sigma_A^2)A_t \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2)s_{g,t} \end{bmatrix} \begin{matrix} (9) \\ (21b) \\ (26) \\ (8) \\ (12) \\ (22) \end{matrix},$$

and the *diffusion function* of the state transition equations:

$$\sigma(\mathbb{Z}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_d d_t & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_A A_t & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} \end{bmatrix} \begin{matrix} (9) \\ (21b) \\ (26) \\ (8) \\ (12) \\ (22) \end{matrix}.$$

By choosing the control $\mathbb{X}_t \in \mathbb{R}_+^2$ at time $t$, the HJB equation reads:

$$\rho V(\mathbb{Z}_t; \mathbb{Y}_t) = \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} \left\{ f(\mathbb{Z}_t, \mathbb{X}_t) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \right\}, \qquad (29)$$

where $V_{\mathbb{Z}}$ is an $n$-vector, $V_{\mathbb{Z}\mathbb{Z}}$ is a $n \times n$ matrix, and $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. A neat result about the formulation of our problem in continuous time is that the HJB equation (29) is, in effect, a deterministic functional equation. In the discrete-time version, we need to numerically approximate expectations (or the $n$-dimensional integral).

The first-order conditions with respect to $c_t$ and $l_t$ for any interior solution are:

$$\frac{d_t}{c_t} = V_a, \tag{30}$$

$$d_t \psi l_t^{\vartheta} = V_a w_t, \tag{31}$$

or, eliminating the costate variable (for $\psi \neq 0$):

$$\psi l_t^{\vartheta} c_t = w_t,$$

which is the standard static optimality condition between labor and consumption.

Most notably, the first-order conditions (30) and (31) yield *optimal controls*:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \equiv \begin{bmatrix} c(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \\ l(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \end{bmatrix} = \begin{bmatrix} (V_a(\mathbb{Z}_t; \mathbb{Y}_t))^{-1} d_t \\ (V_a(\mathbb{Z}_t; \mathbb{Y}_t) w_t / (d_t \psi))^{1/\vartheta} \end{bmatrix}.$$

Thus, the first-order conditions (30) and (31) make the optimal controls functions of the states, $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$, $l_t = l(\mathbb{Z}_t; \mathbb{Y}_t)$. Hence, the concentrated HJB equation reads:

$$\rho V(\mathbb{Z}_t; \mathbb{Y}_t) = f(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t)); \mathbb{Y})^{\top} V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top} V_{\mathbb{Z}\mathbb{Z}}\right). \tag{32}$$

Note that $V_a(\mathbb{Z}_t; \mathbb{Y}_t) = \lambda_t$ in (30) and (31) is readily interpreted as the marginal value of wealth or the current value of a unit of consumption in period $t$, and thus determines the asset pricing kernel in this economy. In what follows, we provide the asset pricing kernel or the stochastic discount factor (SDF) consistent with equilibrium dynamics of macro aggregates, which can be used to price any asset in the economy.

We define the recursive-competitive equilibrium (see Appendix D.3). For the solution, it is instructive to revisit key aggregates. We start from market clearing:

$$c_t = y_t - g_t = (1 - s_g s_{g,t}) y_t = (1 - s_g s_{g,t}) A_t l_t / v_t, \tag{33}$$

such that the combined first-order condition reads:

$$w_t = \psi l_t^{\vartheta} c_t \;\Leftrightarrow\; v_t w_t = \psi l_t^{1+\vartheta} (1 - s_g s_{g,t}) A_t \;\Leftrightarrow\; l_t^{1+\vartheta} = \frac{v_t w_t}{(1 - s_g s_{g,t}) A_t \psi},$$

and from (31):

$$c_t = ((1 - s_g s_{g,t})/v_t)^{\frac{\vartheta}{1+\vartheta}} A_t (mc_t/\psi)^{\frac{1}{1+\vartheta}}, \tag{34}$$

or

$$mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t.$$

For a given level of marginal cost (or wages), the solution is known analytically. In contrast to the flexible-price benchmark, the firms now take into account current marginal cost *and* expected future marginal cost. Hence, the equilibrium value for marginal costs in the sticky-price solution is an unknown function of all states, $mc_t = mc(\mathbb{Z}_t)$.

As we show in the appendix, the marginal value of wealth evolves according to:

$$\begin{aligned} \mathrm{d}\lambda_t &= (\rho - i_t + \pi_t)\lambda_t \mathrm{d}t \\ &\quad + \sigma_d d_t \lambda_d \mathrm{d}B_{d,t} + \sigma_A A_t \lambda_A \mathrm{d}B_{A,t} + \sigma_g s_{g,t} \lambda_g \mathrm{d}B_{g,t} + \sigma_i \lambda_i \mathrm{d}B_{i,t}, \end{aligned} \tag{35}$$

which determines the equilibrium SDF (see Hansen and Scheinkman, 2009):

$$m_s/m_t = e^{-\rho(s-t)} \frac{V_a(\mathbb{Z}_s; \mathbb{Y}_s)}{V_a(\mathbb{Z}_t; \mathbb{Y}_t)}, \quad \text{and} \quad m_t \equiv e^{-\rho t}\lambda_t, \tag{36}$$

or, equivalently, the present value shadow price. Under the risk-neutral measure $\mathbb{Q}$ we may increase the drift of each price process by its covariance with the discount factor, and write a risk-neutral discount factor (see Cochrane, 2005, p.52):

$$\mathrm{d}\lambda_t^{\mathbb{Q}} = (\rho - i_t + \pi_t)\lambda_t^{\mathbb{Q}} \mathrm{d}t. \tag{37}$$

After some algebra (see Appendix D.2), we arrive at the Euler equation, which shows the equilibrium dynamics of consumption:

$$\begin{aligned} \mathrm{d}c_t &= -(\rho - i_t + \pi_t - \sigma_A^2 \tilde{c}_A^2 - \sigma_g^2 \tilde{c}_g^2 - \sigma_i^2 \tilde{c}_i^2 + \rho_d \log d_t + (\tilde{c}_d(1 - \tilde{c}_d) - \tfrac{1}{2})\sigma_d^2) c_t \mathrm{d}t \\ &\quad + \sigma_d \tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A \tilde{c}_A c_t \mathrm{d}B_{A,t} + \sigma_g \tilde{c}_g c_t \mathrm{d}B_{g,t} + \sigma_i \tilde{c}_i c_t \mathrm{d}B_{i,t}, \end{aligned} \tag{38}$$

where $\tilde{c}_i \equiv c_i/c_t$, $\tilde{c}_d \equiv c_d d_t/c_t$, $\tilde{c}_g \equiv c_g s_{g,t}/c_t$, and $\tilde{c}_A \equiv c_A A_t/c_t$, reflecting the slope of the consumption function with respect to the state variables that are driven by shocks.

### 3.3.1. Equilibrium dynamics

So we arrive at a system of 5 endogenous processes, i.e., for the auxiliary variables $x_{1,t}$, $x_{2,t}$, price dispersion $v_t$, the Taylor rule $i_t$, and the consumption Euler equation $c_t$, and 3 exogenous shock processes for $s_{g,t}, d_t, A_t$, which summarize equilibrium dynamics:

$$\begin{aligned}
\mathrm{d}c_t &= -(\rho - i_t + \pi_t - \sigma_A^2\tilde{c}_A^2 - \sigma_g^2\tilde{c}_g^2 - \sigma_i^2\tilde{c}_i^2 + \rho_d \log d_t + (\tilde{c}_d(1-\tilde{c}_d) - \tfrac{1}{2})\sigma_d^2)c_t\mathrm{d}t \\
&\quad +\sigma_d\tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A\tilde{c}_A c_t\mathrm{d}B_{A,t} + \sigma_g\tilde{c}_g c_t\mathrm{d}B_{g,t} + \sigma_i\tilde{c}_i c_t\mathrm{d}B_{i,t} \\
\mathrm{d}x_{1,t} &= ((\rho + \delta + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t \\
\mathrm{d}x_{2,t} &= ((\rho + \delta - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - mc_t d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t \\
\mathrm{d}i_t &= \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\mathrm{d}t + \sigma_i\mathrm{d}B_{i,t} \\
\mathrm{d}v_t &= (\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\,v_t)\,\mathrm{d}t
\end{aligned}$$

in which $(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{\frac{1}{1-\varepsilon}} = \varepsilon/(\varepsilon - 1)(x_{2,t}/x_{1,t})$ determines inflation and

$$\begin{aligned}
d_t/c_t &= ((1 - s_g s_{g,t})/v_t)^{-\frac{\vartheta}{1+\vartheta}}(mc_t/\psi)^{-\frac{1}{1+\vartheta}} d_t/A_t \\
\Leftrightarrow\ mc_t &= \psi((d_t/c_t)(A_t/d_t))^{-(1+\vartheta)}(v_t/(1 - s_g s_{g,t}))^{\vartheta},
\end{aligned} \tag{39}$$

pins down marginal costs. Given a solution to the system of dynamic equations augmented by the stochastic processes (8), (12), and (22), the general equilibrium policy functions (as a function of relevant state variables) can be computed.

### 3.3.2. Numerical solution of the (conditional) deterministic system

In what follows we solve the NK model using the (conditional) deterministic system, which demands that we need to account appropriately for risk. This is obtained if the (nonlinear) solution to the HJB equation implies the same policy function of the boundary value problem. The solution of the deterministic model is contained as a special case.

We start from the HJB equation (32) or the detailed version (D.13) in the web appendix (cf. Appendix D.1), and find that for $V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t) \neq 0$

$$\begin{aligned}
c(\mathbb{Z}_t; \mathbb{Y}_t) &= (i_t - \pi_t)a_t + w_t l(\mathbb{Z}_t; \mathbb{Y}_t) + T_t + F_t - (\rho - (i_t - \pi_t))\frac{V_a(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} \\
&\quad +(\theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_t^*))\frac{V_{ia}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} + \tfrac{1}{2}\sigma_i^2\frac{V_{iia}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} \\
&\quad + \left(\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)v_t\right)\frac{V_{va}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} \\
&\quad -(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2)d_t\frac{V_{da}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} + \tfrac{1}{2}\sigma_d^2 d_t^2\frac{V_{dda}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} \\
&\quad -(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t\frac{V_{Aa}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} + \tfrac{1}{2}\sigma_A^2 A_t^2\frac{V_{AAa}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} \\
&\quad -(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2)s_{g,t}\frac{V_{ga}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2\frac{V_{gga}(\mathbb{Z}_t; \mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t)},
\end{aligned} \tag{40}$$

which we will use to define Euler equation errors.

In what follows, we compute the solution to the HJB equation from a deterministic system of differential equations (a boundary value problem), which works in continuous time since the HJB equation itself becomes a deterministic equation (cf. Chang, 2004).[20] Nevertheless, we have to account appropriately for risk. So the idea is to transform the system of SDEs into a system of PDEs, which also solves the HJB equation. Assume the existence of a consumption function $c_t = c(\mathbb{Z}_t)$, and use Itô's formula to arrive at:

$$\begin{aligned} \mathrm{d}c_t &= c_a \,\mathrm{d}a_t + c_i \,\mathrm{d}i_t + \tfrac{1}{2}c_{ii}\sigma_i^2 \,\mathrm{d}t + c_v \,\mathrm{d}v_t \\ &\quad + c_d \,\mathrm{d}d_t + \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2 \mathrm{d}t + c_A \,\mathrm{d}A_t + \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2 \mathrm{d}t + c_g \,\mathrm{d}s_{g,t} + \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2 \mathrm{d}t. \end{aligned}$$

This leads us to the following proposition.

**Proposition 1.** *By subtracting the Itô second-order terms from the Euler equation (38),*

$$\begin{gathered} dc_t - \tfrac{1}{2}c_{ii}\sigma_i^2 \, dt - \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2 dt - \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2 dt - \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2 dt = \\ c_a \, da_t + c_i \, di_t + c_v \, dv_t + c_A \, dA_t + c_d \, dd_t + c_g \, ds_{g,t}, \end{gathered}$$

*and inserting $dc_t$ from (38) we may eliminate time (and stochastic shocks) and together with $c_t = d_t V_a^{-1}$ yields (40) from the HJB equation.*

**Proof.** Appendix D.4 ■

A system of PDEs which implies the same policy function is constructed using (38) and Proposition 1 by subtracting Itô terms from the Euler equation (accounting for risk) and setting $\mathrm{d}B_{d,t} = \mathrm{d}B_{A,t} = \mathrm{d}B_{g,t} = \mathrm{d}B_{i,t} = 0$ (in the absence of shocks),

$$\begin{aligned} \mathrm{d}c_t &= -(\rho - (i_t - \pi_t))c_t \mathrm{d}t + \tilde{c}_d^2\sigma_d^2 c_t \mathrm{d}t + \tilde{c}_A^2\sigma_A^2 c_t \mathrm{d}t + \tilde{c}_g^2\sigma_g^2 c_t \mathrm{d}t + \tilde{c}_i^2\sigma_i^2 c_t \mathrm{d}t \\ &\quad -\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t \mathrm{d}t - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t \mathrm{d}t - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t \mathrm{d}t - \tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 c_t \,\mathrm{d}t \\ &\quad -c_t\rho_d \log d_t \mathrm{d}t + \tfrac{1}{2}\sigma_d^2 c_t \mathrm{d}t - \tilde{c}_d\sigma_d^2 c_t \mathrm{d}t \end{aligned}$$

where we define $\tilde{c}_{ii} \equiv c_{ii}/c_t$, $\tilde{c}_{dd} \equiv c_{dd}d_t^2/c_t$, $\tilde{c}_{gg} \equiv c_{gg}s_{g,t}^2/c_t$, and $\tilde{c}_{AA} \equiv c_{AA}A_t^2/c_t$ reflecting curvature of the consumption function with respect to the state variables that are driven by shocks, such that $\mathrm{d}c_t = c_a \,\mathrm{d}a_t + c_i \,\mathrm{d}i_t + c_v \,\mathrm{d}v_t + c_A \,\mathrm{d}A_t + c_d \,\mathrm{d}d_t + c_g \,\mathrm{d}s_{g,t}$ solves (40).

[20] In contrast, the discrete-time HJB equation requires the analyst needs to evaluate the state space not only at the current information set, but also at future expected values, so the continuous-time approach does not require to numerically compute expectations (a burdensome step in discrete-time models).

So we refer to the following system of PDEs as the *conditional* deterministic system:

$$\begin{aligned}
\mathrm{d}c_t &= -(\rho-(i_t-\pi_t))c_t\mathrm{d}t+\tilde{c}_d^2\sigma_d^2c_t\mathrm{d}t+\tilde{c}_A^2\sigma_A^2c_t\mathrm{d}t+\tilde{c}_g^2\sigma_g^2c_t\mathrm{d}t+\tilde{c}_i^2\sigma_i^2c_t\mathrm{d}t \\
&\quad -\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2c_t\mathrm{d}t-\tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2c_t\mathrm{d}t-\tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2c_t\mathrm{d}t-\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2c_t\,\mathrm{d}t \\
&\quad -(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2)c_t\mathrm{d}t-\tilde{c}_d\sigma_d^2c_t\mathrm{d}t \qquad (41)\\
\mathrm{d}i_t &= \theta(\phi_\pi(\pi_t-\pi_t^*)+\phi_y(y_t/y_{ss}-1)-(i_t-i_t^*))\,\mathrm{d}t \\
\mathrm{d}v_t &= (\delta\,(1+(1-\varepsilon)(\pi_t-\chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}}+(\varepsilon(\pi_t-\chi\pi_t^*)-\delta)v_t)\,\mathrm{d}t \\
\mathrm{d}d_t &= -\left(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2\right)d_t\mathrm{d}t \\
\mathrm{d}A_t &= -\left(\rho_A\log A_t-\tfrac{1}{2}\sigma_A^2\right)A_t\mathrm{d}t \\
\mathrm{d}s_{g,t} &= -\left(\rho_g\log s_{g,t}-\tfrac{1}{2}\sigma_g^2\right)s_{g,t}\mathrm{d}t \\
\mathrm{d}x_{1,t} &= ((\rho+\delta-(\varepsilon-1)(\pi_t-\chi\pi_t^*))x_{1,t}-d_t/(1-s_gs_{g,t}))\,\mathrm{d}t \\
\mathrm{d}x_{2,t} &= ((\rho+\delta-\varepsilon(\pi_t-\chi\pi_t^*))\,x_{2,t}-mc_td_t/(1-s_gs_{g,t}))\,\mathrm{d}t
\end{aligned}$$

So the Euler equation (41) of the conditional deterministic system is used to obtain the conditional deterministic (or stochastic) steady state.[21] Recall that the inflation rate $\pi_t$ is endogenously determined from (19), and the jump variables $x_{1,t}$ and $x_{2,t}$. We restrict our attention to the solution which leads the economy towards the (stochastic) steady state, in which $\pi_t \to \pi_t^*$. By solving for the time paths, the solution satisfies both the initial and the transversality condition (TVC) and characterizes the stable manifold. We iterate computing controls and updating the derivatives until convergence (cf. Table 1).[22]

Intuitively, the conditional deterministic system summarizes the dynamics under the presence of uncertainty, that agents internalize into their consumption-saving decision, conditional on no further shocks (conditional on the current information set). So agents would not change their (optimal) decision as long as they remain on the stable manifold summarized by the dynamic system, which is idle at the (stochastic) steady state value. Hence, the impulse response functions based on the conditional deterministic system summarize the paths of the stable manifold as implied by the HJB equation.

It is important to note that as long as $\|(\sigma_d,\sigma_A,\sigma_g,\sigma_i)\|\neq 0$, the term $\mathrm{d}c_t$ of the conditional deterministic system (41) does *not* coincide with the term $\mathrm{d}c_t$ of the Euler equation (38), which is an abuse of notation only needed for the numerical solution. Once we derived the policy functions, the original Euler equation is used to simulate the model and/or to make statistical inference, by allowing for the arrival of stochastic shocks.

We solve the system of PDEs by the Waveform Relaxation algorithm. In this way, we can separate the solution in the time dimension from the solution in the policy space, which

[21] Though there will be a steady-state distribution, we follow the convention in the literature and define the fix point of this system as the 'stochastic steady state', and thus use both terms interchangeably.

[22] It is important to note that a recursion as set out in Table 1 is only required if we are interested in the solution of the stochastic model where $\|(\sigma_d,\sigma_A,\sigma_g,\sigma_i)\|\neq 0$.

turns out to be computationally more robust and less expensive. It is possible to parallelize the computation by allocating the grid of state variables to workers. Following the idea in Posch and Trimborn (2013) we obtain the unknown derivatives starting from the solution of the deterministic system, then iteratively define $\tilde{c}_i(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_i/c_t$, $\tilde{c}_{ii}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{ii}/c_t$, $\tilde{c}_d(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_d d_t/c_t$, $\tilde{c}_{dd}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{dd} d_t^2/c_t$, $\tilde{c}_g(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_g s_{g,t}/c_t$, $\tilde{c}_{gg}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{gg} s_{g,t}^2/c_t$, $\tilde{c}_A(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_A A_t/c_t$, and $\tilde{c}_{AA}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{AA} A_t^2/c_t$, and solve the system of ODEs. The initial value for the control and/or jump variables is used to approximate the solution in the policy function space (using tensor products of univariate grids as initial values), then update the solution, and iterate until convergence.

In the boundary value problem (BVP) we seek a function $x : [0,T] \mapsto \mathbb{R}^k$ that satisfies the (conditional) deterministic system consisting of the Euler equation (41) determining $c_t$, and the law of motion for $x_{1,t}, x_{2,t}, i_t, v_t, d_t, A_t$, and $s_{g,t}$ (which gives $k = 8$), together with the given initial conditions for the states $(i_0, v_0, d_0, A_0, s_{g,0})$ and the TVC assuming that variables approach their (stochastic) steady state values. One complication is that the time horizon is infinite, so we use the following transformation of time:

$$t = \frac{\tau}{\nu(1-\tau)} \quad \text{for} \quad \tau \in [0,1),$$

where $\nu$ is a positive (nuisance) parameter, such that for $t \to \infty$ we have that $\tau \to 1$. Alternatively, we may set $T$ sufficiently large but finite number.[23]

### 3.3.3. Numerical solution in the policy function space

We may alternatively solve the HJB equation (32) directly by collocation based on the Matlab `CompEcon` toolbox (Miranda and Fackler 2002).

Since the functional form of the solution is unknown, an alternative strategy for solving the HJB equation is to approximate $V(\mathbb{Z}_t;\mathbb{Y}_t) \approx \phi(\mathbb{Z}_t;\mathbb{Y}_t)v$, in which $v$ is an $n$-vector of coefficients and $\phi$ is the $n \times n$ basis matrix. The computational burden can be reduced when replacing the tensor product by sparse grids (Winschel and Krätzig, 2010). Starting from the HJB equation (32), we may approximation of the value function and/or control variables for given set of collocation nodes and basis functions $\phi(\mathbb{Z}_t;\mathbb{Y}_t)$:

$$\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t)v = f(\mathbb{Z}_t,\mathbb{X}_t) + g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v\right),$$

or

$$v = \left(\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t) - g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}} - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}\right)\right)^{-1} f(\mathbb{Z}_t,\mathbb{X}_t),$$

which yields the coefficients based on a Newton method. This approach, however, requires

[23] Trimborn, Koch, and Steger (2008) introduced the relaxation algorithm to applications in economics. In contrast to their approach, we use projection methods to solve the boundary value problem, which turns out to be relatively efficient and (even for a few approximation points) highly accurate.

a good initial guess, but is extremely useful to verify whether the implied solution obtained from the conditional deterministic system indeed solves the HJB equation.

### 3.3.4. Impulse responses

To compute the impulse response functions (IRFs), we initialize the state variables, given the solution $V(\mathbb{Z}_t; \mathbb{Y}(\mathbb{Z}_t)) \approx \phi(\mathbb{Z}_t; \mathbb{Y}(\mathbb{Z}_t)v$, or the consumption function (40), and solve the resulting system of ODEs following Posch and Trimborn (2013). Because we use a global (and nonlinear) solution technique, in principle, we may initialize the system at any state vector. Hence, we do not need to restrict our analysis to situations, where the economy is assumed to be in the close neighborhood of the steady state (or normal times). This is particularly important since we want to study the equilibrium dynamics in a situation where the nominal interest rate is close to zero and/or the economy is hit by large shocks (non-normal times). In fact, the computed IRF is the equilibrium time path of economic variables, which reflect a single transitional path to the (stochastic) steady state.

## 3.4. Effects of uncertainty and the natural rate of interest

Our results confirm that the effects of uncertainty in the standard NK model are small, and primarily a level shift (cf. Section E.2, and Figures E.16 to E.20 in the web appendix). Both the deterministic and the stochastic steady states are very close.

After having obtained the consumption function (either of the two approaches above) and its derivatives in the nonlinear NK model, we can proceed with Itô's formula and obtain the Euler equation as:

$$\begin{aligned} \mathrm{d}c_t \;=\;& \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\tilde{c}_i c_t \mathrm{d}t + \sigma_i \tilde{c}_i c_t \mathrm{d}B_{i,t} \\ & +(\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\, v_t)c_v \,\mathrm{d}t \\ & - \left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) \tilde{c}_d c_t \mathrm{d}t + \sigma_d \tilde{c}_d c_t \mathrm{d}B_{d,t} \\ & - \left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right) \tilde{c}_A c_t \mathrm{d}t + \sigma_A \tilde{c}_A c_t \mathrm{d}B_{A,t} \\ & - \left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right) \tilde{c}_g c_t \mathrm{d}t + \sigma_g \tilde{c}_g c_t \mathrm{d}B_{g,t} \\ & +\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 c_t \,\mathrm{d}t + \tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t \mathrm{d}t + \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t \mathrm{d}t + \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t \mathrm{d}t \end{aligned} \tag{42}$$

where we used that in general equilibrium $\mathrm{d}a_t = 0$. Using equation (42) together with the Euler equation (38), we are ready to pin down the equilibrium inflation rate as:

$$\begin{aligned}\pi_t &= i_t - \rho + \sigma_A^2\tilde{c}_A^2 + \sigma_g^2\tilde{c}_g^2 + \sigma_i^2\tilde{c}_i^2 + \tilde{c}_d^2\sigma_d^2 - \tilde{c}_d\sigma_d^2 - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) \\ &\quad -\theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\tilde{c}_i \\ &\quad -(\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\, v_t)c_v/c_t \\ &\quad + \left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right)\tilde{c}_d + \left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right)\tilde{c}_A + \left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g \\ &\quad -\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 - \tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2, \end{aligned} \tag{43}$$

to study the effects of uncertainty on the consumption-saving decision and interest rates, compute impulse response functions, and the implied term structure of interest rates.

In general equilibrium, the risk-free rate is defined as (cf. Posch, 2011):

$$\rho - \frac{1}{\mathrm{d}t}\mathbb{E}\left[\frac{\mathrm{d}u'(c_t)}{u'(c_t)}\right] = i_t - \pi_t \equiv r_t^f.$$

Observe that the *implicit* risk premium in the economy is zero, the (instantaneous) return to the government bond is riskless. Even though the zero-coupon bond is default-free, in the general case it is still risky in the sense that its price can covary with the households marginal utility of consumption (Rudebusch and Swanson, 2008, p.115).

In the short-run, only the nominal policy rate is under the control of the monetary authority. Our numerical results show how the equilibrium *real* interest rate $r_t^f$ is affected by the different state variables (cf. Appendix E.2), which together with (43) reads:

$$\begin{aligned} r_t^f &= \rho - \left(\sigma_A^2\tilde{c}_A^2 + \sigma_g^2\tilde{c}_g^2 + \sigma_i^2\tilde{c}_i^2 + \tilde{c}_d^2\sigma_d^2 - \tilde{c}_d\sigma_d^2 - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2)\right) \\ &\quad +\theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\tilde{c}_i \\ &\quad +(\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\, v_t)c_v/c_t \\ &\quad - \left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right)\tilde{c}_d - \left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right)\tilde{c}_A - \left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g \\ &\quad +\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 + \tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 + \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 + \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2. \end{aligned} \tag{44}$$

Following Barsky, Justiniano, and Melosi (2014), we define the natural rate as the real interest rate prevailing in an economy with flexible prices, or the second-best equilibrium (cf. Blanchard and Galí, 2007), which is:

$$\begin{aligned} r_t &= \rho - (\sigma_A^2 + \tilde{c}_g^2\sigma_g^2) + \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 \\ &\quad +\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2 - \left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right) - \left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g \end{aligned} \tag{45}$$

where from (28) we obtain $\tilde{c}_g = -\frac{\vartheta}{1+\vartheta}\frac{s_g}{1-s_g s_{g,t}}s_{g,t}$, and $\tilde{c}_{gg} = -\frac{\vartheta}{(1+\vartheta)^2}\frac{s_g^2}{(1-s_g s_{g,t})^2}s_{g,t}^2$. Defining the (Wicksellian) natural rate as the interest rate once transitory shocks have abated, $r_t^* = \rho - (\sigma_A^2 + \tilde{c}_g^2\sigma_g^2) + \frac{1}{2}\tilde{c}_{gg}\sigma_g^2$, we shed light on potential sources of shocks to the natural rate (temporary or permanent). An increase in uncertainty of either technology or fiscal policy shocks depresses the natural rate. These results are complementary to Barsky, Justiniano, and Melosi (2014), who find that increases in patience, i.e., declines in $\rho$ (often referred to as discount factor shocks), lower the natural rate $r_t^*$ and are isomorphic to higher uncertainty about future productivity. Indeed, such uncertainty shocks provide an attractive structural interpretation for the identified permanent shocks.

Our analysis clearly shows that the spread between the equilibrium risk-free rate and the interest rate under certainty is affected by staggered price setting. For example, while $\tilde{c}_A = 1$ and $\tilde{c}_{AA} = 0$ in the frictionless limit, the staggered price equilibrium has $\tilde{c}_A \approx 0.57$ and $\tilde{c}_{AA} = 0.28$ (evaluated at the stochastic steady-state). So while uncertainty about future technology depresses the *natural rate* relative to the interest rate under certainty by about 4 bp, uncertainty does increase the (long-run) real interest rate in the NK model by about 3 bp (for the parameterization see Table 2).[24]

The definition of the natural rate is consistent with the economy operating at its full potential, i.e., the output that would have prevailed in an flexible-price economy:

$$y_t^n = (1 - s_g s_{g,t})^{-\frac{1}{1+\vartheta}} A_t((\varepsilon - 1)/(\varepsilon\psi))^{\frac{1}{1+\vartheta}}. \tag{46}$$

Based on the definition, the output gap is readily available from $x_t \equiv y_t/y_t^n - 1$.

### 3.5. Term structure of interest rates and the term premium

Following Rudebusch and Swanson (2012), the term premium on long-term nominal bonds compensates investors for inflation and consumption risk over the lifetime of the bond. The term premium can be defined by comparing the equilibrium price under the physical and the risk-neutral probability measure. Consider a (zero-coupon) bond with unity payoff at maturity $N$. Using the expectation approach, the equilibrium price reads:

$$P_t^{(N)} = \mathbb{E}_t \left( m_{t+N}/m_t e^{-\int_t^{t+N} \pi_s ds} \right), \tag{47}$$

which from (35) and (36) can be solved, by simulating for a given maturity $N$:

$$\begin{aligned} P_t^{(N)} \quad = \quad & \mathbb{E}_t \Big( e^{-\int_t^{t+N} \left( r_s^f + \pi_s + \frac{1}{2}\sigma_d^2 (1-\tilde{c}_d)^2 + \frac{1}{2}\sigma_A^2 \tilde{c}_A^2 + \frac{1}{2}\sigma_g^2 \tilde{c}_g^2 + \frac{1}{2}\tilde{c}_i^2 \sigma_i^2 \right) \mathrm{d}s} \\ & \times e^{\int_t^{t+N} \sigma_d (1-\tilde{c}_d) \mathrm{d}B_{d,s} - \sigma_A \tilde{c}_A \mathrm{d}B_{A,s} - \sigma_g \tilde{c}_g / \lambda_s \mathrm{d}B_{g,s} - \tilde{c}_i \sigma_i \mathrm{d}B_{i,s}} \Big) \end{aligned}$$

forward a few thousand times, and take the average. Similarly, we obtain the hypothetical bond price under the risk-neutral probability measure, i.e., using a risk-neutral discount

[24] Observe that the negative effect on the equilibrium risk-free rate in the flexible-price scenario compares to the effects on $r$ in the endowment economy (cf. Posch, 2011, Corollary 2.1).

factor (or equivalently the risk-free rate) rather than the household's SDF:

$$\begin{aligned} \tilde{P}_t^{(N)} &= \mathbb{E}_t^{\mathbb{Q}}\left(m_{t+N}/m_t e^{-\int_t^{t+N}\pi_s ds}\right) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+N} i_s\,\mathrm{d}s}\right) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+N}(r_s^f+\pi_s)\,\mathrm{d}s}\right), \end{aligned} \tag{48}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation under the risk-neutral probability measure and thus the risk-neutral evaluation of the bond price (Rudebusch and Swanson, 2008, 2012). Then, the yield is the (fictional) interest rate that justifies the quoted price, such that the log price $p_t^{(N)} \equiv \log P_t^{(N)}$ satisfies $y_t^{(N)} = -(1/N)p_t^{(N)}$, and the yield curve is a plot of the yields as a function of their maturity. Hence, the expectation approach is easily adapted for the perfect-foresight solution (the term premium would be zero).

The expectation approach is quite common in macroeconomic models of the term structure (Gürkaynak, Sack, and Swanson, 2005; Rudebusch and Swanson, 2008, 2012; Andreasen, Fernández-Villaverde, and Rubio-Ramírez, 2018). One drawback is that the computation of the whole term structure is quite challenging. It requires the simulation in the time dimension to study the resulting $N$-periods ahead distribution of $P_t^{(N)}$, for a given *particular* state. Naturally, it puts a limit to study the long-end of the yield curve, which therefore cannot easily be explored.

An alternative solution offers the PDE approach (Cochrane, 2005, chap. 19.4), in which the basic pricing equation for the price $P_t^{(N)}$ reads:

$$\mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)})/P_t^{(N)}\right) - \left(1/P_t^{(N)}(\partial P_t^{(N)}/\partial N) + i_t\right)\,\mathrm{d}t = -\mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)}/P_t^{(N)})\,(\,\mathrm{d}\lambda_t/\lambda_t)\right),$$

or for the price under the risk-neutral probability measure $\mathbb{Q}$

$$\mathbb{E}_t^{\mathbb{Q}}\left((\,\mathrm{d}\tilde{P}_t^{(N)})/\tilde{P}_t^{(N)}\right) - \left(1/\tilde{P}_t^{(N)}(\partial \tilde{P}_t^{(N)}/\partial N) + i_t\right)\,\mathrm{d}t = 0.$$

Observe that in equilibrium, the prices $P_t^{(N)}$ and $\tilde{P}_t^{(N)}$ are functions of the state variables, so we may write $P_t^{(N)} = P^{(N)}(\mathbb{Z}_t)$, and obtain the PDE:

$$\begin{aligned} &\theta(\phi_\pi(\pi_t-\pi_t^*) + \phi_y(y_t/y_{ss}-1) - (i_t-i_t^*))(\partial P_t^{(N)}/\partial i_t) + \tfrac{1}{2}\sigma_i^2(\partial^2 P_t^{(N)}/(\partial i_t)^2) \\ &+(\delta(1-(\varepsilon-1)(\pi_t-\chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t-\chi\pi_t^*)-\delta)v_t)(\partial P_t^{(N)}/\partial v_t) \\ &-\left(\rho_d\log d_t - \tfrac{1}{2}\sigma_d^2\right)d_t(\partial P_t^{(N)}/\partial d_t) + \tfrac{1}{2}\sigma_d^2 d_t^2(\partial^2 P_t^{(N)}/(\partial d_t)^2) \\ &-\left(\rho_A\log A_t - \tfrac{1}{2}\sigma_A^2\right)A_t(\partial P_t^{(N)}/\partial A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2(\partial^2 P_t^{(N)}/(\partial A_t)^2) \\ &-\left(\rho_g\log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right)s_{g,t}(\partial P_t^{(N)}/\partial s_{g,t}) + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2(\partial^2 P_t^{(N)}/(\partial s_{g,t})^2) \\ &-(\partial P_t^{(N)}/\partial N) - i_t P_t^{(N)} = -\sigma_d^2 d_t(1-\tilde{c}_d)(\partial P_t^{(N)}/\partial d_t) \\ &+\sigma_i^2\tilde{c}_i(\partial P_t^{(N)}/\partial i_t) + \sigma_A^2 A_t\tilde{c}_A(\partial P_t^{(N)}/\partial A_t) + \sigma_g^2 s_{g,t}\tilde{c}_g(\partial P_t^{(N)}/\partial s_{g,t}). \end{aligned} \tag{49}$$

where for the hypothetical price $\tilde{P}_t^{(N)}$, the covariance terms on the RHS are zero.

The solution to the pricing equation implies the complete term structure of interest rate for *any* given state vector and maturity,

$$y_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) = -\log P^{(N)}(\mathbb{Z}_t)/N, \qquad \tilde{y}_t^{(N)} \equiv \tilde{y}^{(N)}(\mathbb{Z}_t) = -\log \tilde{P}^{(N)}(\mathbb{Z}_t)/N, \tag{50}$$

and the term premium

$$TP_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - \tilde{y}^{(N)}(\mathbb{Z}_t), \tag{51}$$

which measures the 'riskiness' of the bonds. The premium reflects the compensation that investors require for bearing the risk that $r_t^f$ does not evolve as expected.

Our strategy for solving the PDE (49) is to approximate the function $P_t^{(N)} \approx \Phi(N, \mathbb{Z}_t)v$ in which $v$ is the vector of coefficients and $\Phi$ denotes the basis matrix (alternatively we may use finite differences as in Achdou, Han, Lasry, Lions, and Moll, 2017). For illustration, Figure 17 shows the yield curves (50) following a negative shock to the natural rate. Here, the difference between the physical and the risk-neutral probability measure defines the term premium (51), which is about 6 bp (10 bp using the feedback rule).

### 3.6. ZIRP period revisited

Let us now reconsider the interest rate shock together with shocks to the natural rate. The new insights we get are really due to the nonlinearities. First, we simulate the 2007Q4 to 2011Q3 interest rate shock of −475 bp (and shock to the Wicksellian rate of −150 bp) together with a temporary 'preference shock' of −15 percent, in which $d_t$ is assumed now to follow the logistic process (cf. Appendix A.2 and Figure E.44 in the web appendix):

$$\mathrm{d}d_t = \rho_d (d_t - \bar{d})\,(1 - d_t)\,/(1 - \bar{d})\,\mathrm{d}t, \quad d_t > \bar{d}, \tag{52}$$

with $\bar{d} = 0.79$ and $\rho_d = 0.4214$.[25] It implies that the initial value ($d_0 = 0.846$) is only 5 percentage points above the lower bound. In other words, this shock to the natural rate is 'large' such that it will have different dynamics than small shocks. This particular parameterization has been chosen simply to illustrate that the implied interest rate process (of the full nonlinear approach) now is a prolonged period of an *apparently* binding ZLB with negative natural rates, but this ZIRP episode has an active monetary policy.

Second, we revisit the inflation rebound in 2011Q3 with interest rates immobile at zero. With a policy rate close to zero in 2010Q4 (0.20%) and zero inflation target, we simulate a shock to the inflation target of 200 bp back to the announced target rate together with a large negative shock to the natural rate (a shock to the Wicksellian rate of −150 bp, and a temporary 'preference shock' of −15 percent). Again, the thought experiment implies

[25] Note that with the assumed logistic process for the preference shock the Euler equation (38) changes for the nonlinear model (cf. Section D.2).

an inflation rebound consistent with the dynamics of key macroeconomic aggregates, the yield curves *and* at the same time a ZIRP period of 8 quarters (cf. Figures 15 and 16).

Note that the three-equation model shows quite similar dynamics as the nonlinear version, but fails to capture the nonlinear effects of the logistic process. This result is because the dynamics of the linear model (6) are the same as (52) only for small shocks. Why have researchers not studied dynamics beyond the OU process (6) so far? The local dynamics of the linear approximation could have been quite successful in 'normal times' with smaller shocks. So the (unobserved) shocks might have been well described by simple OU processes. In fact, the local dynamics are observationally equivalent to the dynamics of the OU process (cf. Appendix A.2). But the model dynamics can be quite different in non-normal times. Moreover, we show that such shocks must be large in order to drag the interest rate close to (potentially below) zero values. Because the traditional local approximation schemes would be inappropriate for large shocks, we confirm Brunnermeier and Sannikov (2014) that nonlinearities can be important in times of crises.

So distinguishing between normal and non-normal times, in which the dynamics are different from those at the steady state, is one alternative interpretation in which a *single* shock to the natural rate generates the ZIRP period. We show that (52) is a parsimonious specification to capture the dynamics of large negative shocks (non-normal times).

### 3.7. Discussion of the new insights

The full (nonlinear) approach and the local dynamics give rise to at least three insights. First, uncertainty of shocks has an effect on the natural rate and is one potential structural explanation for permanent shocks. The effects of risk are negligible in the NK model without additional frictions.[26] Second, the PDE approach is a promising alternative to traditional approaches when computing the term structure of interest rates consistent with equilibrium dynamics of macro aggregates. We show that both the nominal and real yield curves provide useful information about expectations, term premia, and for the identification of shocks.[27] Third, the parsimonious three-equation NK model does not inherit important nonlinearities (compare Figures 2 and E.14), but a nonlinear shock dynamics can easily generate a ZIRP period with a single shock (see Figure 15).

## 4. Conclusion

In this paper we show the ability of the NK model to explain the recent episodes. We find that temporary and permanent shocks to the interest rate (and inflation) are required

[26]This result is not surprising, but effects of risk can be quantitatively important in models with habit formation and capital adjustment costs (e.g., Parra-Alvarez, Polattimur, and Posch, 2020).

[27]Introducing recursive preferences produces a large and variable term premium without compromising the model's ability to fit key macroeconomic variables (cf. Rudebusch and Swanson, 2012).

to replicate the dynamics of key macroeconomic aggregates consistently with the term structure of interest rates. We show that this NK model supports both views, either higher interest rates result into higher long-run inflation (neo-Fisherian view), or higher interest can temporarily reduce inflation (traditional view). One potential interpretation is that monetary policy actions (changes in policy rates) may trigger variations in target rates, a view that is motivated by empirical data (Uribe, 2017). Allowing for temporary and permanent shocks to the natural rate allows us to understand several puzzles in the literature, including apparent term structure anomalies. We also show that in a nonlinear approach a single shock can generate a ZIRP period with stable and quiet inflation, fully consistent with the model predictions. This paper is the first to provide a full analytical investigation of the effects of uncertainty in the continuous-time NK model, which sheds light on the natural rate. Our results confirm that uncertainty shocks are isomorphic to discount factor shocks (Barsky, Justiniano, and Melosi, 2014), so they provide an attractive structural interpretation of permanent shocks to the natural rate.

We believe that this paper is a starting point for several lines of research. First, our benchmark specification is useful for the comparison with a medium-scale model, allowing for other nominal and/or real frictions, habit formation, variable capacity utilization and adjustment cost as in models used by central banks for policy analysis, and/or including a financial sector (e.g., Brunnermeier and Sannikov, 2014). Second, the structural parameters should be estimated from empirical data. Though our identification of shocks provides a good benchmark, eventually the data should be used to jointly pin down the structural parameters together with the shocks. One key advantage of the continuous-time approach is that the model solution is consistent with different frequencies of macro and financial data (cf. Christensen, Posch, and van der Wel, 2016). Here, two promising alternatives are either to apply the continuous-time econometric toolbox from the financial literature to our macroeconomic (or macro-finance) models, or to combine the discrete-time estimation approaches with an Euler discretization scheme of the equilibrium dynamics. Third, we should study the monetary policy transmission in a heterogeneous-agent economy, e.g., with idiosyncratic income shocks (cf. Kaplan, Moll, and Violante, 2018).

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# A. Appendix

## A.1. New Keynesian analysis

This section sheds some light on the implications of the NK model for both the IS-curve and the NK Phillips-curve. For illustration, we abstract from the effects of uncertainty by setting variance terms to zero to compare the solution of the nonlinear model with the three-equation model (cf. Figures 2).[28]

We start with the NK forward-looking Phillips-curve, which from (20), the first-order condition $\lambda_t = d_t/c_t$, and the market-clearing condition $c_t = (1 - s_g s_{g,t})y_t$ reads:

$$\mathrm{d}(\pi_t - \chi\pi_t^*) = -(\delta - (\varepsilon - 1)(\pi_t - \chi\pi_t^*))\left(\pi_t - \chi\pi_t^* + (mc_t/x_{2,t} - 1/x_{1,t})\, d_t/(1 - s_g s_{g,t})\right)\mathrm{d}t,$$

which together with $mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t$ and $l_t = y_t v_t/A_t$, among other variables, shows the response of inflation to the output gap.[29] Hence, the linearized Phillips-curve around deterministic steady-state values reads (see Section D.5 for definitions)

$$\begin{aligned}
\mathrm{d}(\pi_t - \chi\pi_t^*) \;=\;& -a_2(\pi_t - \chi\pi_t^* - (1-\chi)\pi_{ss})\mathrm{d}t \\
& -a_2\left(\rho + \delta - \varepsilon(1-\chi)\pi_{ss}\right)(mc_t/mc_{ss} - 1)\mathrm{d}t \\
& -a_2\left(\rho + \delta + (1-\varepsilon)(1-\chi)\pi_{ss}\right)(x_{1,t}/x_{1,ss} - 1)\mathrm{d}t \\
& +a_2\left(\rho + \delta - \varepsilon(1-\chi)\pi_{ss}\right)(x_{2,t}/x_{2,ss} - 1)\mathrm{d}t \\
& +a_2(1-\chi)\pi_{ss}(d_t/d_{ss} - 1)\mathrm{d}t \\
& +a_2(1-\chi)\pi_{ss}\left(s_g s_{g,ss}/(1 - s_g s_{g,ss})\right)(s_{g,t}/s_{g,ss} - 1)\mathrm{d}t
\end{aligned}$$

in which $a_2 \equiv \delta + (1-\varepsilon)(1-\chi)\pi_{ss}$ and

$$\pi_t - \chi\pi_t^* - (1-\chi)\pi_{ss} = a_2(x_{2,t}/x_{2,ss} - x_{1,t}/x_{1,ss})$$

It shows in the NK Phillips-curve how the change in inflation depends on marginal costs. We may insert the linearized equation for marginal cost,

$$\begin{aligned}
mc_t/mc_{ss} - 1 \;=\;& (1+\vartheta)(y_t/y_{ss} - 1) - (1+\vartheta)(A_t/A_{ss} - 1) \\
& +\vartheta(v_t/v_{ss} - 1) - s_g s_{g,ss}/(1 - s_g s_{g,ss})(s_{g,t}/s_{g,ss} - 1) \\
=\;& (1+\vartheta)(c_t/c_{ss} - 1) - (1+\vartheta)(A_t/A_{ss} - 1) \\
& +\vartheta(v_t/v_{ss} - 1) + \vartheta(s_g s_{g,ss}/(1 - s_g s_{g,ss}))(s_{g,t}/s_{g,ss} - 1)
\end{aligned}$$

[28] In the web appendix we compare to the solution of the stochastic model (cf. Section E.2)

[29] In order to analyze local dynamics, the traditional approach is to (log-)linearize the variables. We define $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$, where $x_{ss}$ is the steady-state value for the variable $x_t$, such that $x_t = (1+\hat{x}_t)x_{ss}$

where

$$y_t/y_{ss} = c_t/c_{ss} + (s_g s_{g,ss}/(1 - s_g s_{g,ss}))(s_{g,t}/s_{g,ss} - 1)$$

to obtain the NK Phillips-curve with respect to output and/or consumption. Moreover,

$$\begin{aligned} \mathrm{d}(v_t/v_{ss} - 1) &= \frac{\varepsilon(1-\chi)\pi_{ss}}{\delta + (1-\varepsilon)(1-\chi)\pi_{ss}}(\pi_t - \chi\pi_t^* - (1-\chi)\pi_{ss})\mathrm{d}t \\ &\quad + (\varepsilon(1-\chi)\pi_{ss} - \delta)(v_t/v_{ss} - 1)\mathrm{d}t \end{aligned}$$

From (38), the linearized Euler equation reads:

$$\begin{aligned} \mathrm{d}(c_t/c_{ss} - 1) &= (i_t - i_t^* - (\pi_t - \pi_t^*) - \rho_d(d_t/d_{ss} - 1))\mathrm{d}t \\ &= (i_t - \rho - \pi_t - \rho_d(d_t/d_{ss} - 1))\mathrm{d}t \end{aligned}$$

which is readily interpreted as the (micro-founded) NK IS-curve. For comparison with the literature, in the case without technology and government expenditure shocks, this is Werning's (2012) continuous-time specification:

$$\mathrm{d}(c_t/c_{ss} - 1) = (i_t - r_t - \pi_t)\mathrm{d}t$$

and the natural rate reads $r_t \equiv \rho + \rho_d(d_t/d_{ss} - 1))$.

To summarize, the equilibrium dynamics of the linearized system can be simplified in the version with full price indexation ($\chi = 1$) to:

$$\begin{aligned} \mathrm{d}(c_t/c_{ss} - 1) &= (i_t - \rho - \pi_t - \rho_d(d_t/d_{ss} - 1))\mathrm{d}t \\ \mathrm{d}\pi_t &= (\rho(\pi_t - \pi_t^*) - \delta\,(\rho+\delta)\,(mc_t/mc_{ss} - 1) + \delta(d_t/d_{ss} - 1))\mathrm{d}t \\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_t^*))\mathrm{d}t \\ \mathrm{d}(v_t/v_{ss} - 1) &= -\delta(v_t/v_{ss} - 1)\mathrm{d}t \end{aligned}$$

After transitional dynamics and by assuming $s_g = 0$, this system coincides with the model in Werning (2012) and Cochrane (2017b) :

$$\begin{aligned} \mathrm{d}x_t &= (i_t - r_t - \pi_t)\mathrm{d}t \\ \mathrm{d}\pi_t &= (\rho(\pi_t - \pi_t^*) - \delta\,(\rho+\delta)\,(1+\vartheta)x_t)\,\mathrm{d}t \\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y x_t - \theta(i_t - i_t^*))\mathrm{d}t \end{aligned}$$

where $x_t \equiv c_t/c_{ss} - 1$ defines the output gap. Solving forward finally yields:

$$\pi_t - \pi_t^* = \int_t^\infty e^{-\rho(s-t)}\delta(\rho+\delta)(1+\vartheta)x_s\mathrm{d}s \tag{A.1}$$

### A.1.1. Determinacy

While the three-equation NK model with a feedback rule introduces the interest rate as a control variable, the partial adjustment model makes the interest rate a state variable, which is given by past inflation. For the ease of presentation, we set $r_t = r_t^* = \rho$.

The simple NK model with a *feedback rule* has no relevant state variables. The system can be analyzed in terms of two equations (1) and (2) using (3a). A unique locally bounded solution requires two positive eigenvalues of the Jacobian matrix[30]

$$A_1 = \begin{bmatrix} 0 & \phi - 1 \\ -\kappa & \rho \end{bmatrix}.$$

Hence, a necessary (and sufficient) condition for local determinacy is $\phi > 1$. So the unique locally bounded solution is $x_t = 0$ and $\pi_t = \pi_t^*$ such that $i_t = \rho + \pi_t^*$. In other words, a negative (short-run) response of inflation to raising interest rates is not possible as long as the monetary authority implements the Taylor principle. Any monetary policy shock, which affects the policy targets, would be permanent and operates instantaneously. The response of inflation is unambiguously *positive*. In this perfect-foresight model, interest rates can be expressed in terms of future output gaps. We would also need to include a serially correlated shock in order to generate transitional dynamics in the model.

In the simple NK model with *partial adjustment*, the only relevant state variable is the interest rate (historically given inflation rates). We thus obtain the equilibrium values for the output gap and the inflation rate as policy functions $x_t = x(i_t)$ and $\pi_t = \pi(i_t)$. The system can be analyzed in terms of three equations (1), (2) and (3b), where a unique locally bounded solution requires two positive eigenvalues of the Jacobian matrix[31]

$$A_2 = \begin{bmatrix} 0 & -1 & 1 \\ -\kappa & \rho & 0 \\ 0 & \phi\theta & -\theta \end{bmatrix}.$$

Again, a necessary (and sufficient) condition for local determinacy is $\phi > 1$. One caveat is that the model is linearized around zero inflation targets (or full indexation). It can be shown that the condition $\phi > 1$ remains necessary (for details see web appendix).

[30] The Jacobian matrix has $\mathrm{tr}(A_1) = \lambda_1 + \lambda_2 = \rho > 0$ and $\det(A_1) = (\phi - 1)\kappa$ is positive for $\phi > 1$, thus both eigenvalues have positive real parts, $\lambda_1\lambda_2 = \det(A_1)$, such that $\lambda_{1,2} = \frac{1}{2}(\rho \pm \sqrt{\rho^2 - 4((\phi - 1)\kappa)})$.

[31] Note that $\det(A_2) = -\kappa\theta(\phi - 1)$ which is negative for $\phi > 1$. Further, we know that $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A_2) = \rho - \theta$ and $\lambda_1\lambda_2\lambda_3 = \det(A_2) = -\kappa\theta(\phi - 1)$. Because a unique locally bounded solution requires two positive eigenvalues, $\phi > 1$ is necessary (and sufficient) to obtain determinacy in this model.

## A.2. Alternative shock dynamics

Consider an alternative specification of a logistic growth process:

$$\mathrm{d}d_t = \rho_d d_t \left(1 - d_t\right) \mathrm{d}t \tag{A.2}$$

which is the logistic growth model with carrying capacity 1. The natural (lower) bound is zero and the turning point is 0.5. If the variable is near its carrying capacity, the dynamics are just like those of the process in (8), whereas if the variable is near its lower bound, the dynamics are similar to exponential growth. An extended version, such that it fits our needs to have a prolonged period of persistence of a shock at the beginning and later to revert back to the steady state level geometrically at rate $\rho_d$ such that the higher $\rho_d$ the lower persistence, the smaller $\rho_d$ the more pronounced shocks are smeared out in time.

Now consider

$$\mathrm{d}d_t = \rho_d (d_t - \bar{d}) \left(1 - d_t\right) / (1 - \bar{d}) \, \mathrm{d}t \tag{A.3}$$

of which the solution is

$$d_t = \frac{d_{ss} - \bar{d}}{1 + \mathbb{C} e^{-\rho_d t}} + \bar{d}.$$

The (unique) steady state value is the solution of

$$0 = \rho_d \left(d_t - \bar{d}\right) \left(1 - \left(d_t - \bar{d}\right) / (d_{ss} - \bar{d})\right) \, \mathrm{d}t$$

where we require that $d_t > \bar{d}$ for all time. Linearizing about $d_{ss}$ yields

$$\mathrm{d}d_t / d_{ss} = -\rho_d (d_t / d_{ss} - 1) \, \mathrm{d}t \quad \Leftrightarrow \quad \mathrm{d}\hat{d}_t = -\rho_d \hat{d}_t \, \mathrm{d}t.$$

It reflects that the logistic growth model for $d_t - \bar{d}$ such that $d_t$ approaches $d_{ss}$. The variable $d_t - \bar{d}$ is defined on 0 and $\infty$ with carrying capacity $d_{ss} - \bar{d}$ and turning point at $(d_{ss} - \bar{d})/2$, such that the original variable $d_t$ is defined between $\bar{d}$ and $\infty$ with turning point at $1 - (d_{ss} - \bar{d})/2$. For $\bar{d} = 0$ we assume logistic growth for $d_t$, whereas $\bar{d} \to d_{ss}$ squeezes the admissible region lower than the steady state level towards zero, such that $\bar{d}$ denotes the lower bound for $d_t$. Any (negative) shock larger than $1 - (d_{ss} - \bar{d})/2$ induces completely different dynamics, staying there for some time before returning to the steady state level (cf. Figure E.44). This effect only shows up in the nonlinear version of the model. While the logistic model looks very much like an exponential model in the beginning, around the steady state value, the linearized dynamics are the same as for the Ornstein-Uhlenbeck process. Hence, the linear model (6) would *not* capture those dynamics.

# B. Tables and Figures

## B.1. Tables

Table 1: Summary of the solution algorithm

| | | |
|---|---|---|
| Step 1 | (Initialization) | Provide an initial guess for the unknown derivatives for a given set of collocation nodes and basis functions. |
| Step 2 | (Solution) | Compute the optimal value of the controls for the set of nodal values for the state variables. |
| Step 3 | (Update) | Update the consumption function derivatives. |
| Step 4 | (Iteration) | Repeat Steps 2 and 3 until convergence. |

Table 2: Parameterization

| | | |
|---|---|---|
| $\vartheta$ | 1 | Frisch labor supply elasticity |
| $\rho$ | 0.03 | subjective rate of time preference, $\rho = -4 \log 0.9925$ |
| $\psi$ | 1 | preference for leisure |
| $\delta$ | 0.65 | Calvo parameter for probability of firms receiving signal, $\delta = -4 \log 0.85$ |
| $\varepsilon$ | 25 | elasticity of substitution intermediate goods |
| $s_g$ | 0 | share of government consumption |
| $\rho_d$ | 0.4214 | autoregressive component preference shock, $\rho_d = -4 \log 0.9$ |
| $\rho_A$ | 0.4214 | autoregressive component technology shock, $\rho_A = -4 \log 0.9$ |
| $\rho_g$ | 0.4214 | autoregressive component government shock, $\rho_g = -4 \log 0.9$ |
| $\sigma_d$ | 0.02 | variance preference shock |
| $\sigma_A$ | 0.02 | variance technology shock |
| $\sigma_g$ | 0 | variance government shock |
| $\sigma_i$ | 0.02 | variance monetary policy shock |
| $\phi_\pi$ | 4 | inflation response Taylor rule |
| $\phi_y$ | 0 | output response Taylor rule |
| $\theta$ | 0.5 | interest rate response Taylor rule |
| $\pi_{ss}$ | 0.02 | inflation target rate |
| $\chi$ | 1 | indexation at steady-state inflation rate, $\chi = 1$ is full indexation |

## B.2. Figures

Figure 1: US federal funds rate, 10-year treasury rate and inflation rate
In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index (Core CPI), seasonally adjusted, the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), at the monthly frequency, and the Output gap (HP Filter) at the quarterly frequency. All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January, 1990, through August, 2020.

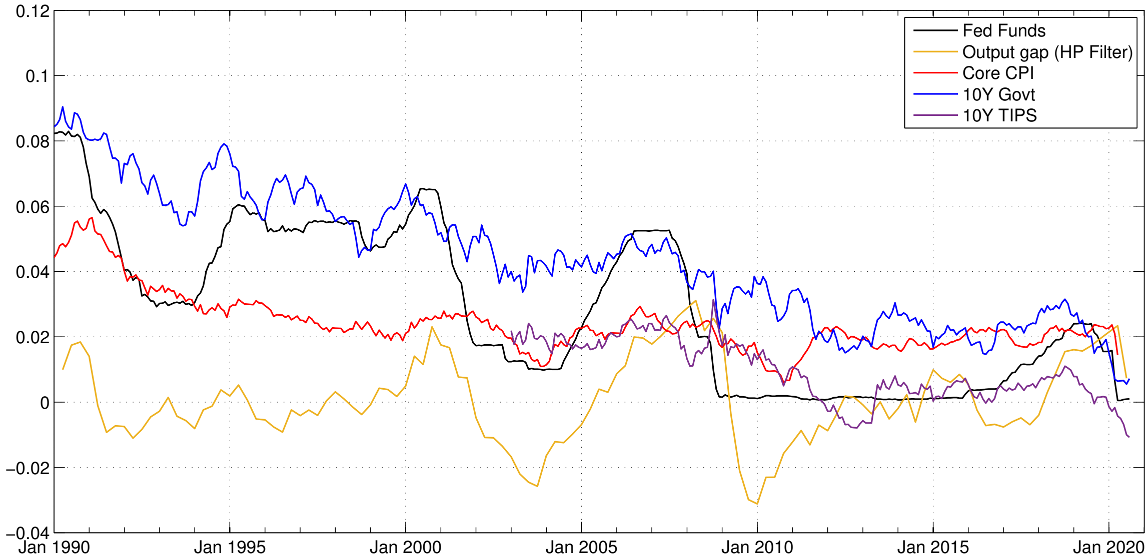


Figure 2: Solution of the three-equation NK model with partial adjustment
In this figure we show (from left to right) the output gap, and the inflation rate as a function of the (initial) interest rate for a parameterization $(\rho, \kappa, \phi, \theta, \pi_t^*) = (0.03, 0.8842, 4, 0.5, 0.02)$.

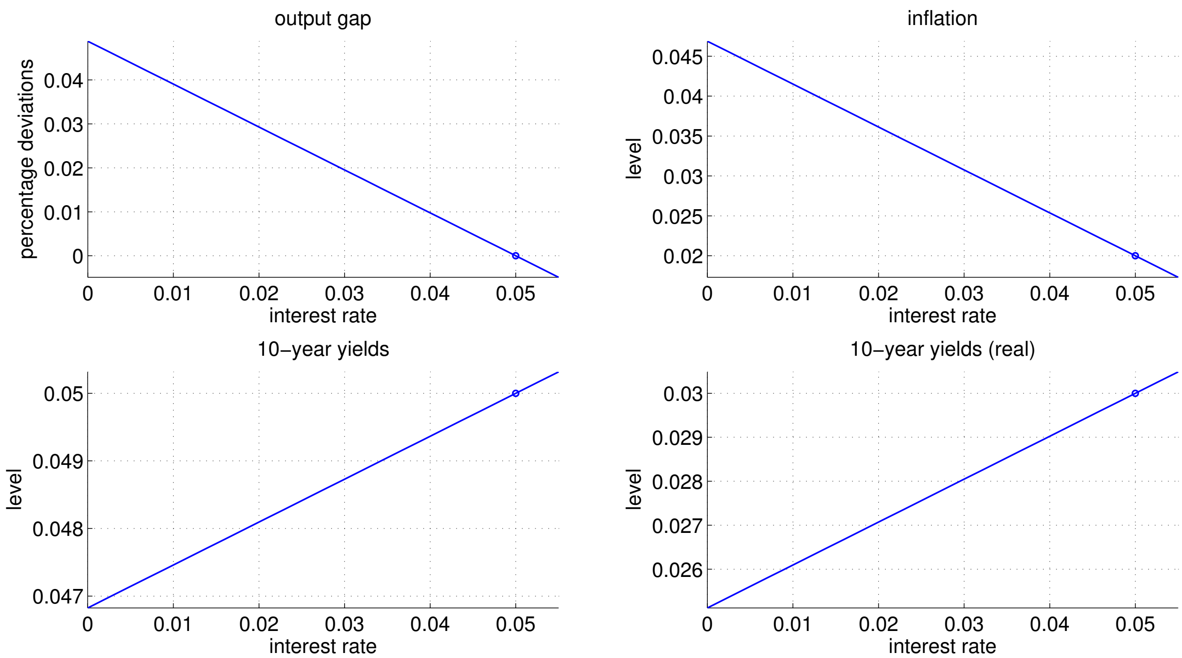

Figure 3: Implied natural rate

In this figure we show time series plots of the model-implied natural rate using the simple NK model with temporary shocks to the natural rate, by matching the monthly US Effective Federal Funds Rate (Fed Funds) and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency. The sample runs from January, 1990, through August, 2020.

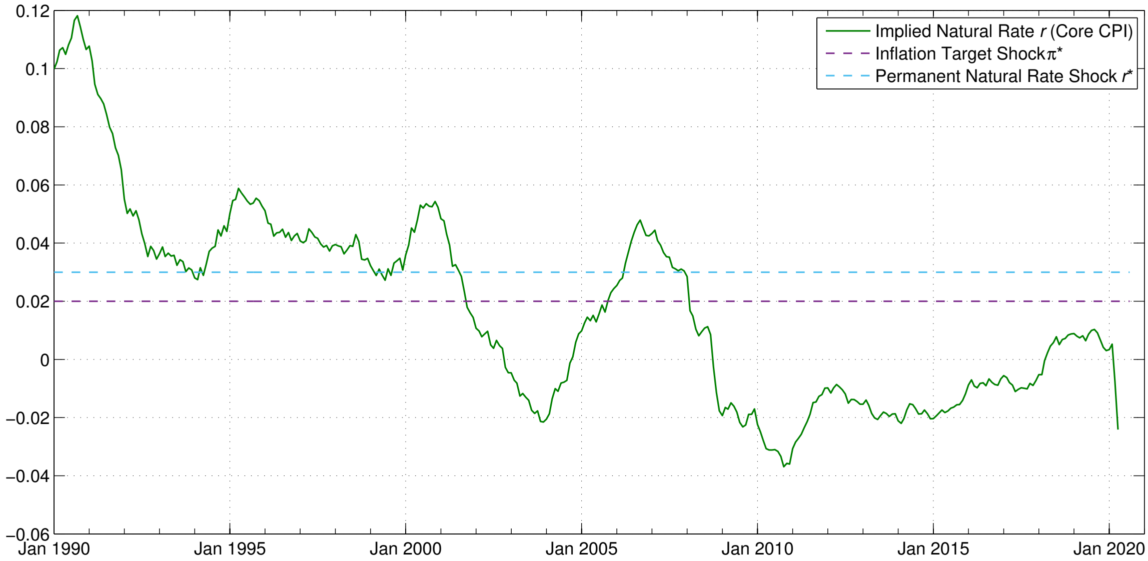


Figure 4: Implied natural rate

In this figure we show time series plots of the model-implied natural rate using the simple NK model with temporary shocks to the natural rate, by matching the quarterly US Effective Federal Funds Rate (Fed Funds) and minimizing the distance to the Consumer Price Index (Core CPI), seasonally adjusted, and the Output gap (HP Filter) at the quarterly frequency from 1990Q1 through 2020Q2.

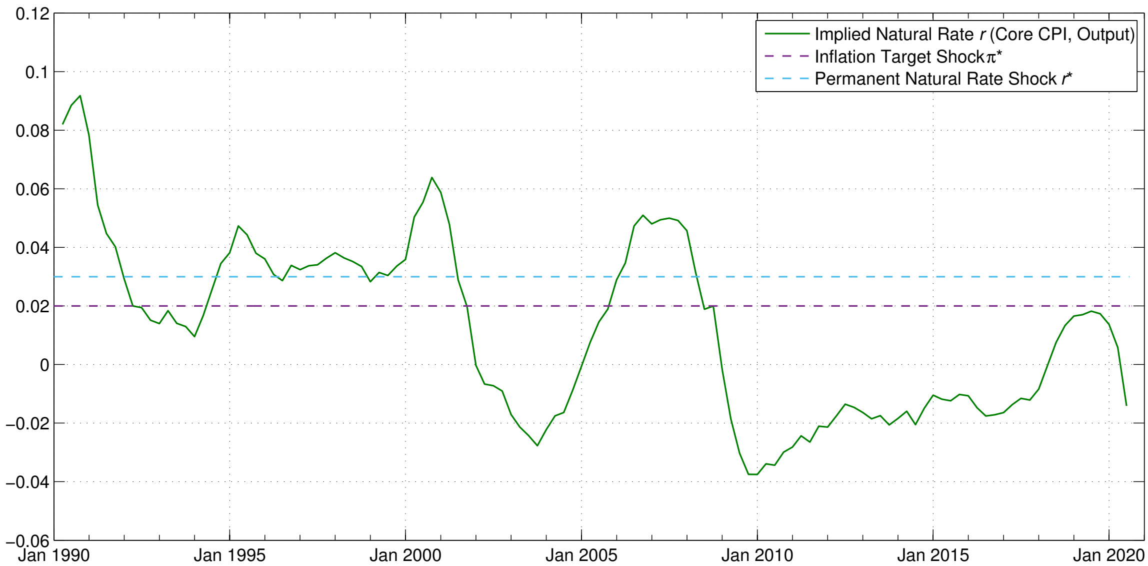

Figure 5: Implied inflation rates and 10-year treasury rates

In this figure we show time series plots of the model-implied inflation and the 10-year treasury rates using the simple NK model with temporary shocks to the natural rate, by matching the monthly US Effective Federal Funds Rate (Fed Funds), and minimizing the distance to the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency, from January, 1990, through August, 2020.

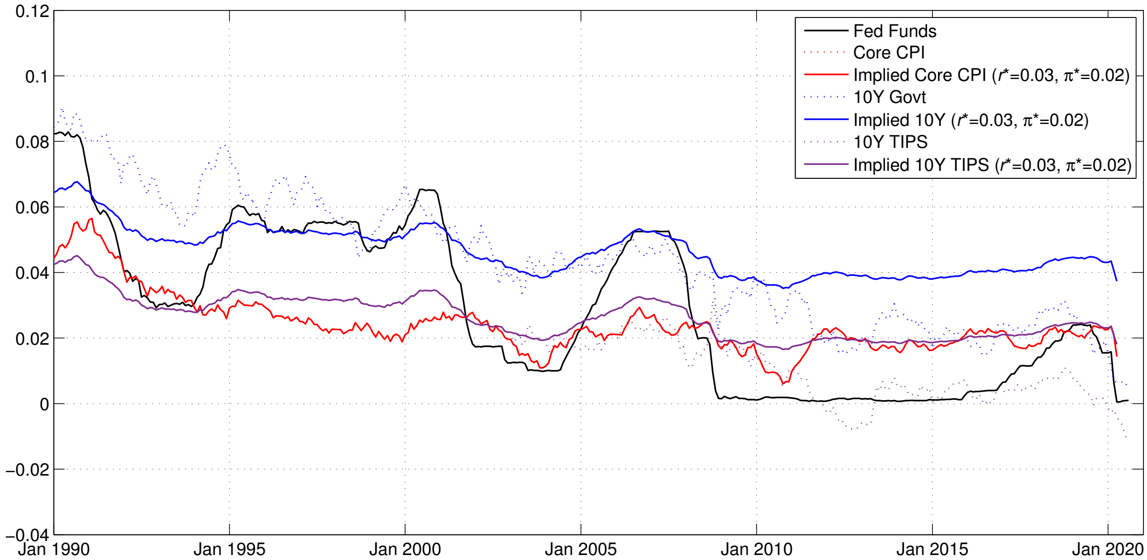


Figure 6: Implied inflation rates, 10-year treasury rates and output gap

In this figure we show time series plots of the model-implied inflation, 10-year treasury rates, and the output gap using the simple NK model with temporary shocks to the natural rate, matching the quarterly US Effective Federal Funds Rate (Fed Funds) and minimizing the distance to the Consumer Price Index (Core CPI), seasonally adjusted, and the Output gap (HP Filter) from 1990Q1 through 2020Q2.

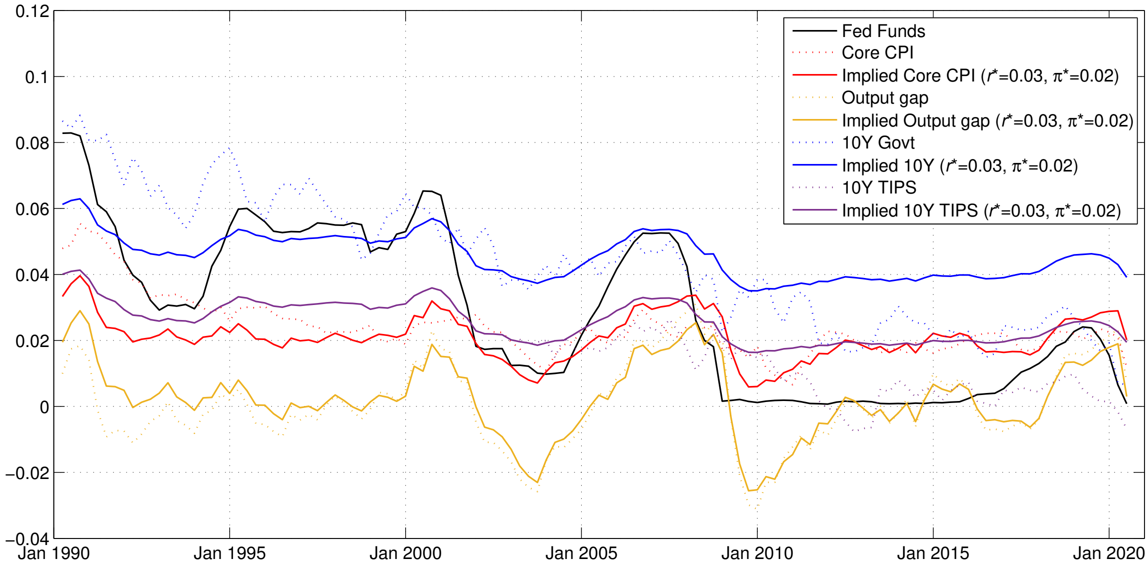

Figure 7: Implied natural rate

In this figure we show time series plots of the model-implied natural rate using the simple NK model with temporary and permanent shocks to the natural rate and inflation, by matching the monthly US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency. Restricted by data availability of 10Y TIPS, the sample runs from January, 2003, through August, 2020.

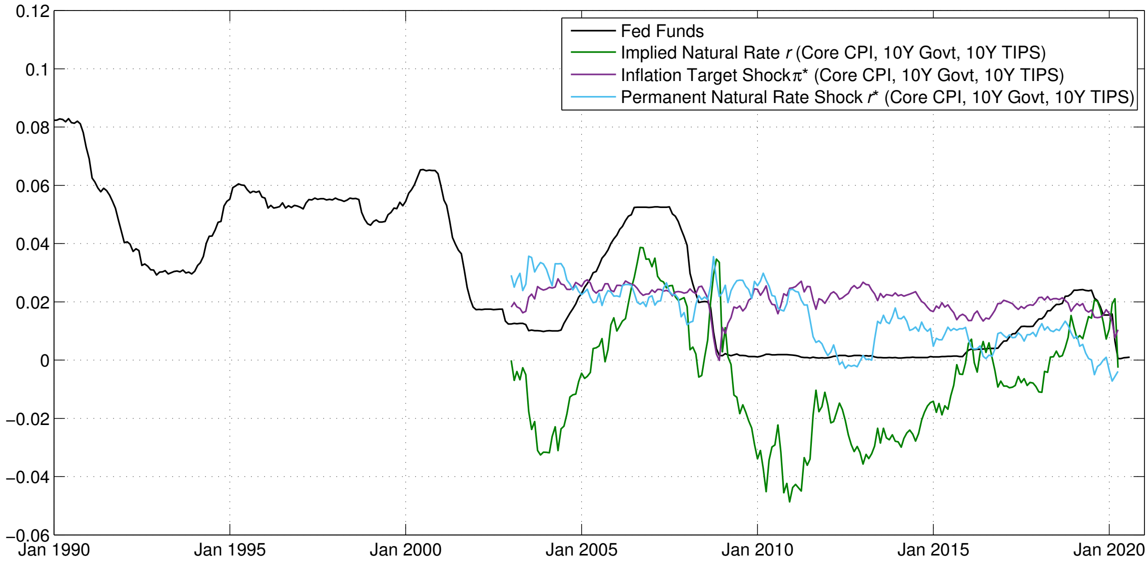


Figure 8: Implied natural rate

In this figure we show time series plots of the model-implied natural rate using the simple NK model with temporary and permanent shocks to the natural rate and inflation, by matching the quarterly US Effective Federal Funds Rate (Fed Funds), and the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), the Consumer Price Index (Core CPI), seasonally adjusted, and the Output gap (HP Filter) at the quarterly frequency. Restricted by data availability of 10Y TIPS, the sample runs from 2003Q1 through 2020Q2.

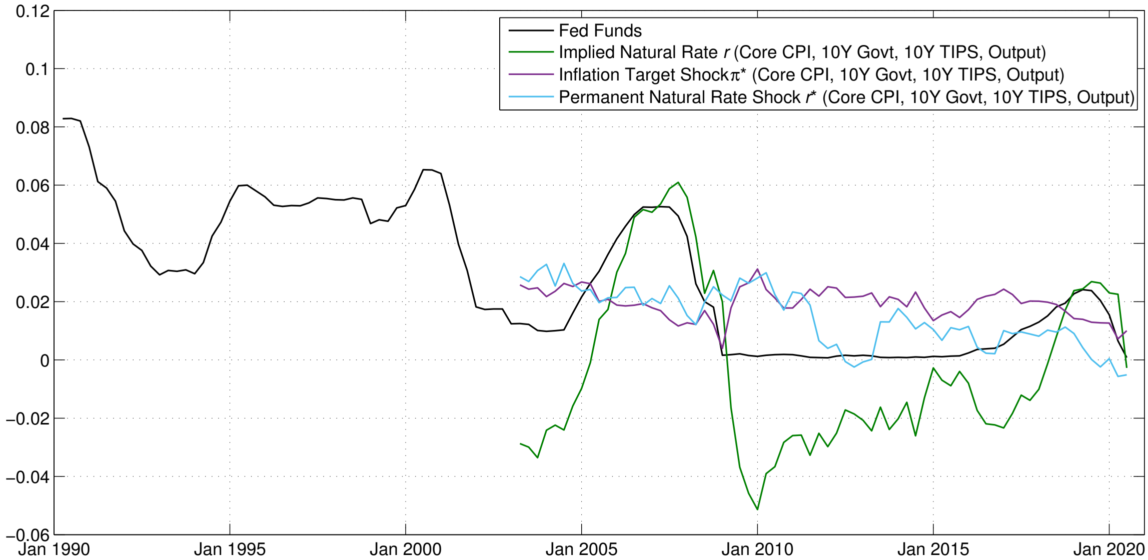

Figure 9: Implied inflation rates and 10-year treasury rates
In this figure we show time series plots of the model-implied inflation and the 10-year treasury rates using the simple NK model with temporary and permanent shocks to the natural rate and inflation, by matching the monthly US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency. The sample runs from January, 2003, through August, 2020.

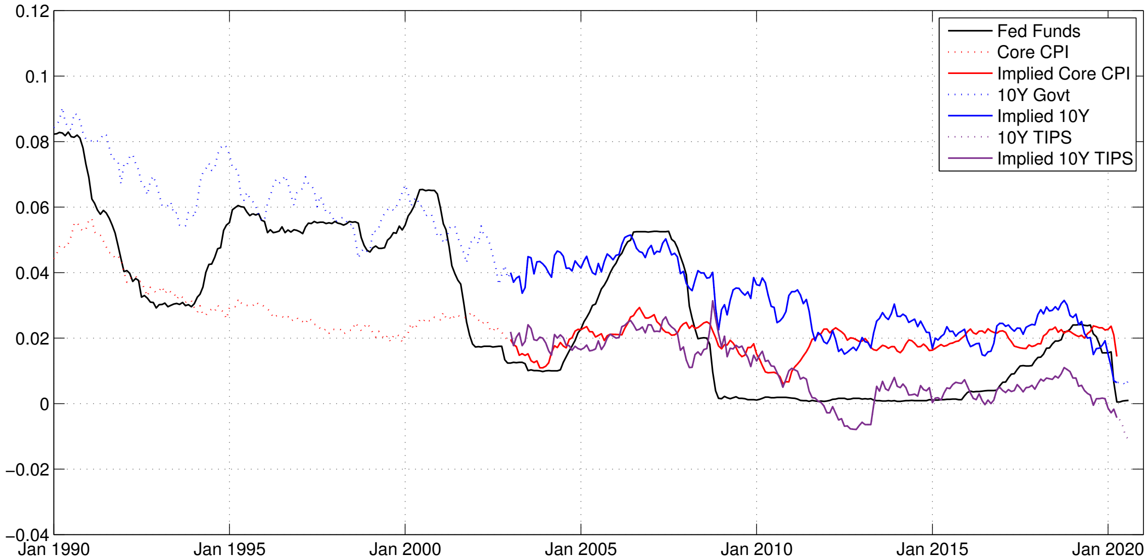


Figure 10: Implied inflation rates, 10-year treasury rates and output gap
In this figure we show time series plots of the model-implied inflation, 10-year treasury rates, and the output gap using the simple NK model with temporary and permanent shocks to the natural rate and inflation, by matching the quarterly US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Consumer Price Index (Core CPI), seasonally adjusted, and the Output gap (HP Filter) at the quarterly frequency from 2003Q1 through 2020Q2.

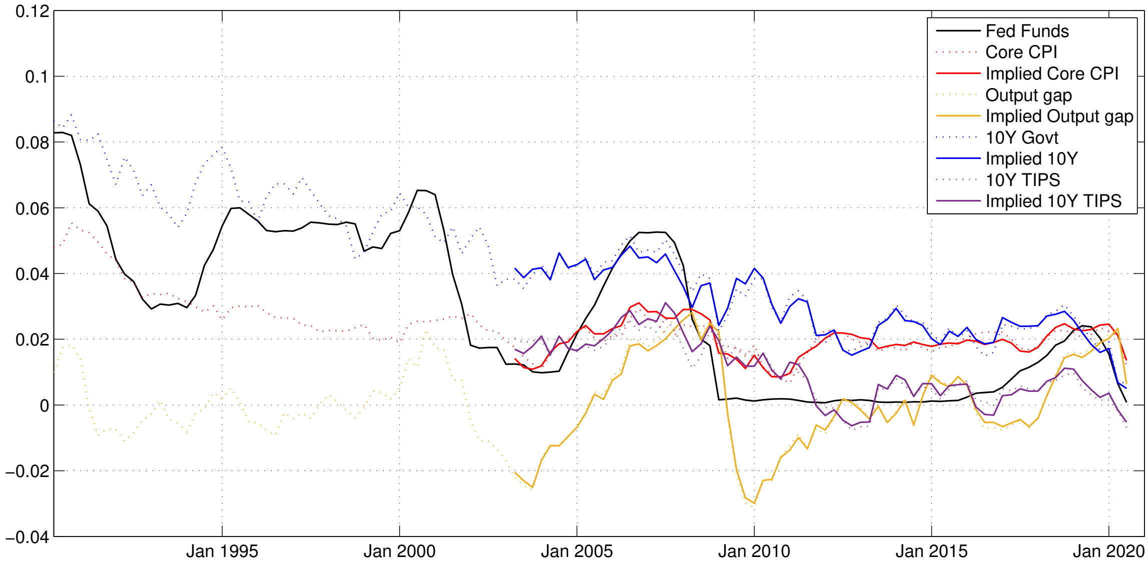

Figure 11: Simulated responses to identified shocks (2001-2003)

In this figure we show (from left to right, top to bottom) the simulated responses to the identified shocks (cf. Figures 8 and E.9 in the web appendix for 2000Q4), with effects for the output gap, the inflation rate, the interest rate, and the 10-year yields (blue solid), and the pre-shock scenario (dotted); predicted initial values (circle) and data (cross).

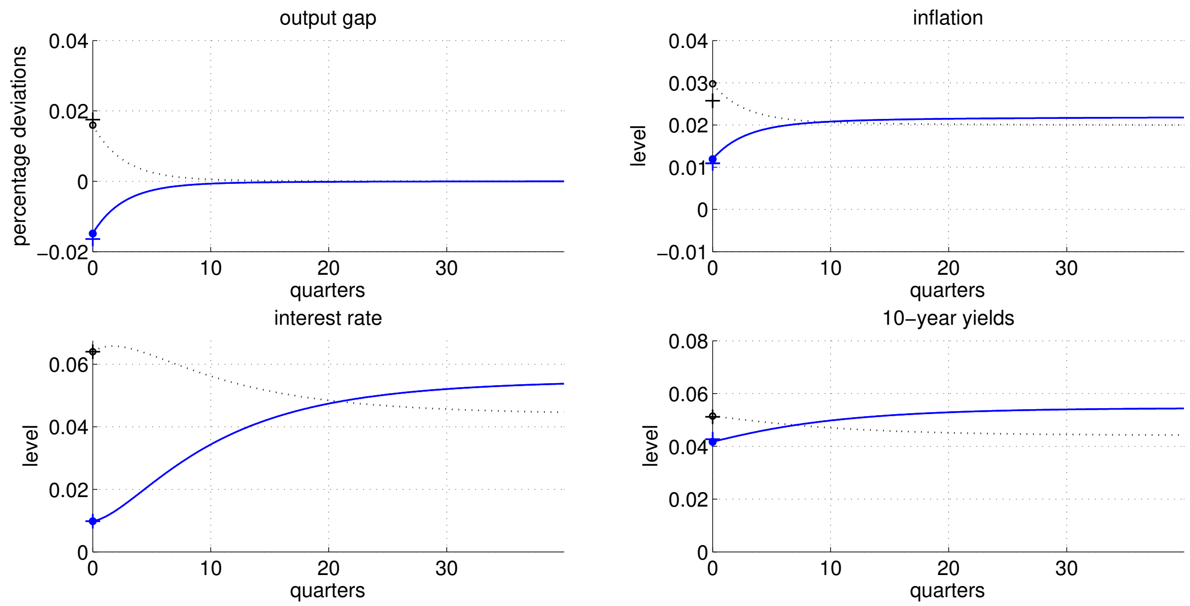


Figure 12: Implied yield curves for the identified shocks (2001-2003)

In this figure we show (from left to right, top to bottom) the implied yield curve for the identified shocks (cf. Figures 8 and E.9 until 2002Q4), with effects for the nominal and real yields (blue solid), and the pre-shock scenario (dotted); observed yields are indicated with a cross (TIPS are available from 2003Q1)

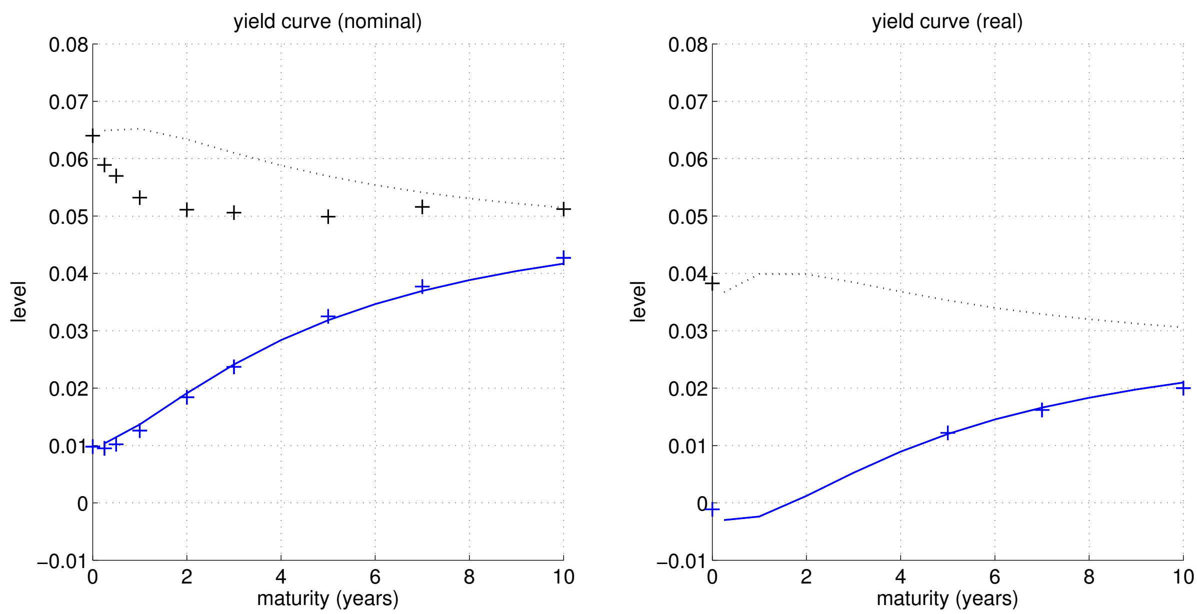

Figure 13: Simulated responses to identified shocks (2004-2005)
In this figure we show (from left to right, top to bottom) the simulated responses to the identified shocks (cf. Figure 8), with effects for the output gap, the inflation rate, the interest rate, and the 10-year yields (blue solid), and the pre-shock scenario (dotted); predicted initial values (circle) and data (cross).

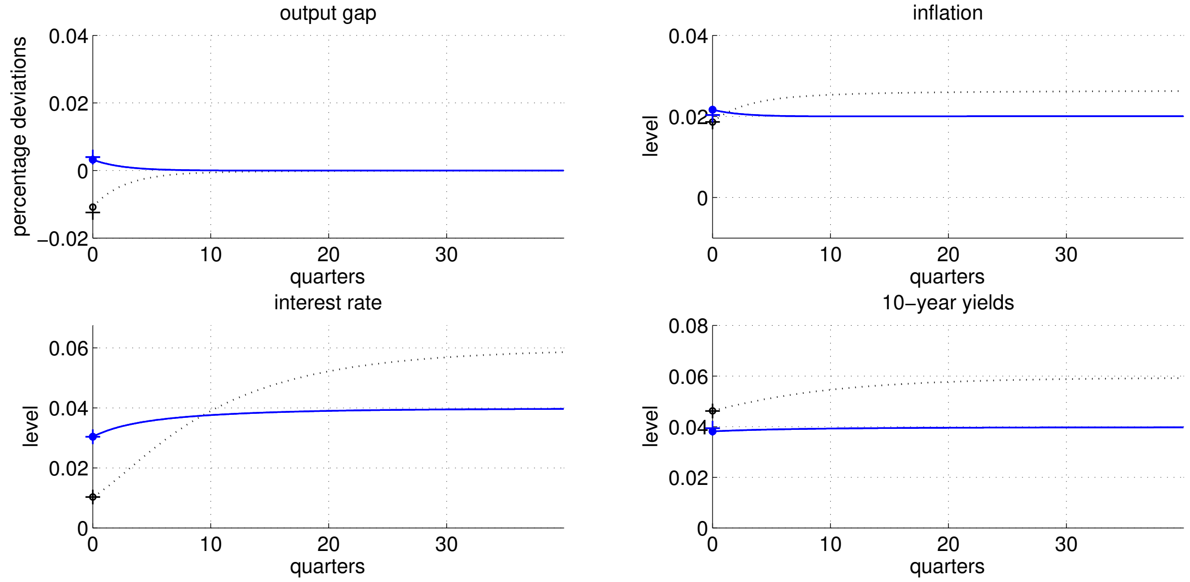


Figure 14: Implied yield curve for the identified shocks (2004-2005)
In this figure we show (from left to right, top to bottom) the implied yield curve for the identified shocks (cf. Figure 8), with effects for the nominal and real yields (blue solid), and the pre-shock scenario (dotted); observed yields are indicated with a cross.

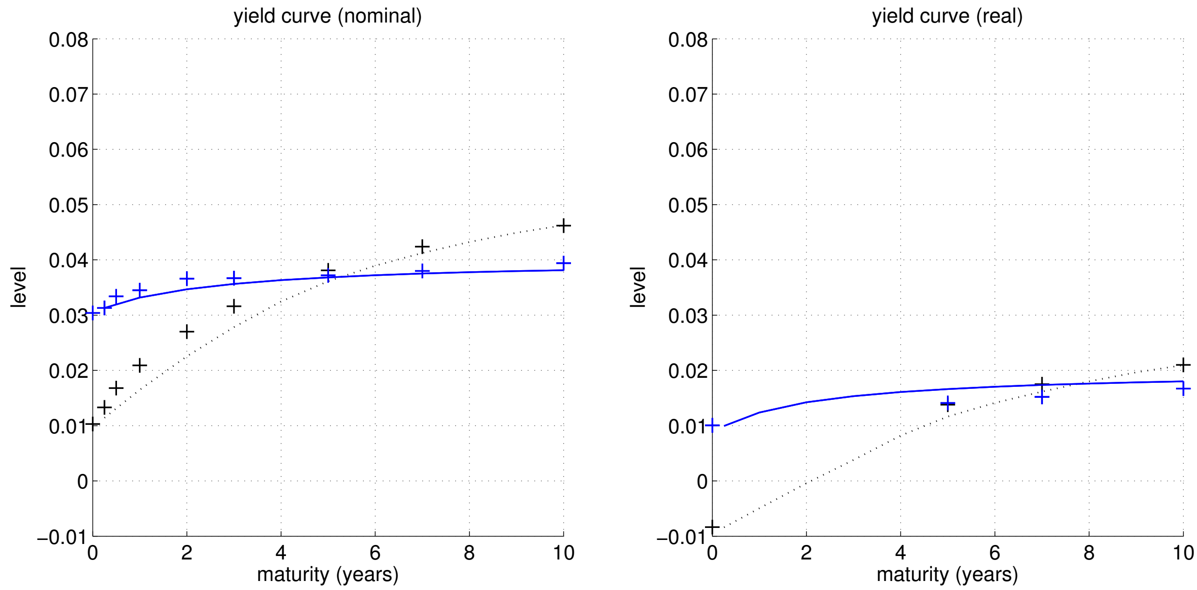

Figure 15: Simulated responses to hypothetical shocks (2010-2011)
In this figure we show (from left to right, top to bottom) the simulated responses to unexpected shocks to the inflation target rate (0.02), the Wicksellian rate (−0.015), the logistic process ($\bar{d} = 0.79$) for preferences (−0.15), and its effect on the output gap, the inflation rate, and the level/slope of the interest rate (blue solid), the no-natural rate shock scenario (black dashed, $r_t^* = 0.03$, $\pi_t^* = 0.02$), and the no-target shock scenario (dotted, $r_t^* = 0.03$, $\pi_t^* = 0$); predicted initial values (circle) and data (cross).

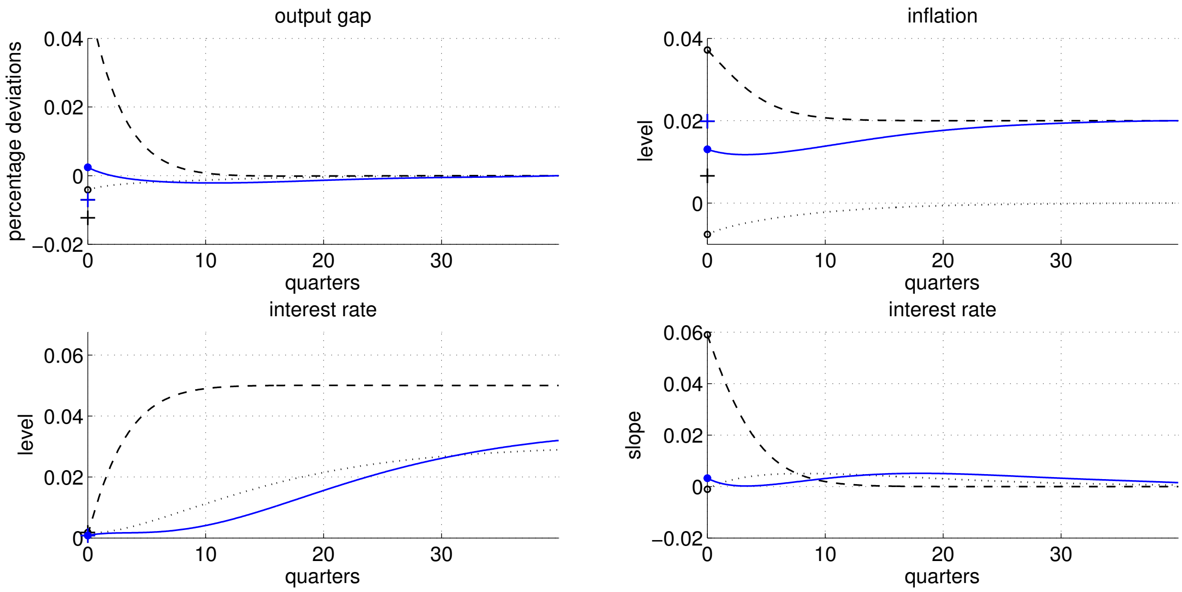


Figure 16: Implied yield curve for the hypothetical shocks (2010-2011)
In this figure we show the yield curve response to unexpected shocks to the inflation target rate (0.02), the Wicksellian rate (−0.015), the logistic process ($\bar{d} = 0.79$) for preferences (−0.15), with effects for the nominal and real yields (blue solid), no-natural rate shock scenario (black dashed, $r_t^* = 0.03$, $\pi_t^* = 0.02$), and the no-target shock scenario (dotted, $r_t^* = 0.03$, $\pi_t^* = 0$); observed yields are indicated with a cross.

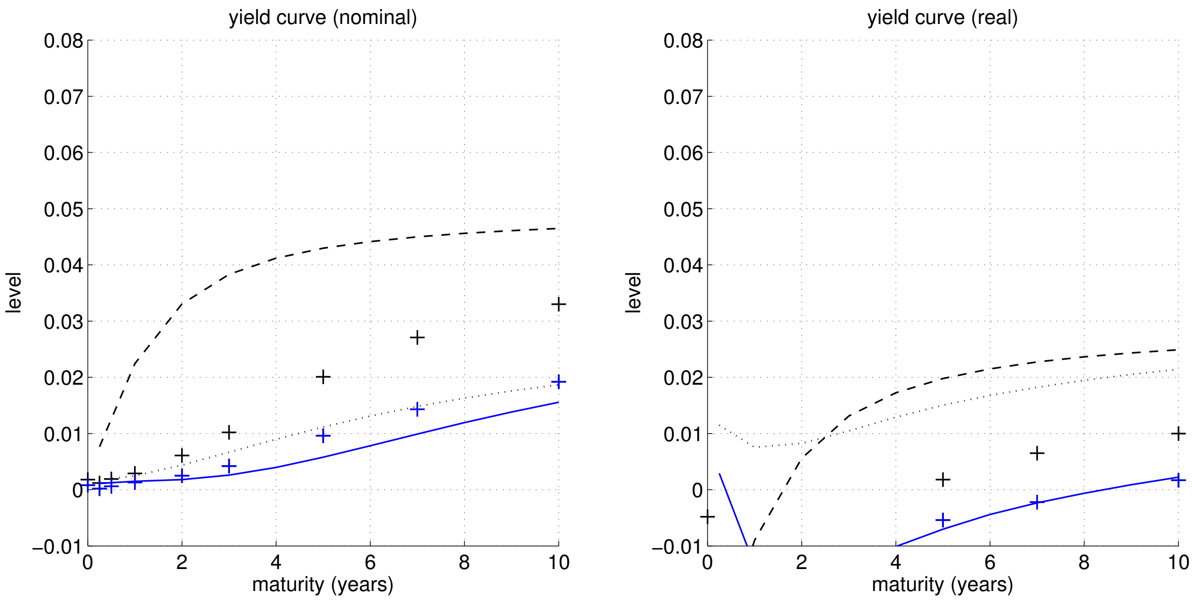

Figure 17: Term structure of interest rates in the stochastic NK model
In this figure we show (from left to right, top to bottom) the nominal and the real term structure of interest rates in the partial adjustment model (blue), and with a feedback rule value (red) after an unexpected shock to preferences (−0.1). The dashed lines show the risk-neutral term structure.